THE RICCI TENSOR OF AN ALMOST HOMOGENOUS KÄHLER MANIFOLD

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Abstract. We determine an explicit expression for the Ricci tensor of a K-manifold, that is of a compact Kähler manifold M with vanishing first Betti number, on which a semisimple group G of biholomorphic isometries acts with an orbit of codimension one. We also prove that the Kähler form ω and the Ricci form ρ of M are uniquely determined by two special curves with values in $\mathfrak{g} = Lie(G)$, say $Z_\omega, Z_\rho : \mathbb{R} \to \mathfrak{g} = Lie(\mathfrak{g})$ and we show how the curve Z_ρ is determined by the curve Z_ω.

These results are used in another work with F. Podestà, where new examples of non-homogeneous compact Kähler-Einstein manifolds with positive first Chern class are constructed.

1. Introduction.

The objects of our study are the so-called *K-manifolds*, that is Kähler manifolds (M, J, g) with $b_1(M) = 0$ and which are acted on by a group G of biholomorphic isometries, with regular orbits of codimension one. Note that since M is compact and G has orbits of codimension one, the complexified group $G^{\mathbb{C}}$ acts naturally on M as a group of biholomorphic transformations, with an open and dense orbit. According to a terminology introduced by A. Huckleberry and D. Snow in [HS], M is *almost-homogeneous* with respect to the $G^{\mathbb{C}}$-action. By the results in [HS], the subset $S \subset M$ of singular points for the $G^{\mathbb{C}}$-action is either connected or with exactly two connected components. If the first case occurs, we will say that M is a *non-standard K-manifold*; we will call it *standard K-manifold* in the other case.

The aim of this paper is furnish an explicit expression for the Ricci curvature tensor of a K-manifold, to be used for constructing (and possibly classify) new families of examples of non-homogeneous K-manifold with special curvature conditions. A successful application of our results is given in [PS1], where several new examples of non-homogeneous compact Kähler-Einstein manifolds with positive first Chern class are found.

Note that explicit expressions for the Ricci tensor of standard K-manifolds can be found also in [Sa], [KS], [PS] and [DW]. However our results can be applied to any kind of K-manifold and hence they turn out to be particularly useful for the

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non-standard cases (at this regard, see also [CG]). They can be resumed in the following three facts.

Let $\mathfrak{g}$ be the Lie algebra of the compact group G acting on the K-manifold (M, J, g) with at least one orbit of codimension one. By a result of [PS1], we may always assume that G is semisimple. Let also $\mathcal{B}$ be the Cartan-Killing form of $\mathfrak{g}$. Then for any x in the regular point set M_{reg}, one can consider the following $\mathcal{B}$-orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \ , \tag{1.1}$$

where $\mathfrak{l} = \mathfrak{g}_x$ is the isotropy subalgebra, $\mathbb{R}Z + \mathfrak{m}$ is naturally identified with the tangent space $T_o(G/L) \simeq T_x(G \cdot x)$ of the G-orbit $G/L = G \cdot x$, and $\mathfrak{m}$ is naturally identified with the holomorphic subspace $\mathfrak{m} \simeq \mathcal{D}_x$

$$\mathcal{D}_x = \{ \ v \in T_x(G \cdot x) \ : \ Jv \in T_x(G \cdot x) \ \} \ . \tag{1.2}$$

Notice that for any point $x \in M_{\text{reg}}$ the $\mathcal{B}$-orthogonal decomposition (1.1) is uniquely given; on the other hand, two distinct points $x, x' \in M_{\text{reg}}$ may determine two distinct decompositions of type (1.1).

Our first result consists in proving that any K-manifold has a family $\mathcal{O}$ of smooth curves $\eta : \mathbb{R} \to M$ of the form

$$\eta_t = \exp(itZ) \cdot x_o \ ,$$

where $Z \in \mathfrak{g}$, $x_o \in M$ is a regular point for the $G^{\mathbb{C}}$-action and the following properties are satisfied:

(1) η_t intersects any regular G-orbit;
(2) for any point $\eta_t \in M_{\text{reg}}$, the tangent vector η'_t is transversal to the regular orbit $G \cdot \eta_t$;
(3) any element $g \in G$ which belongs to a stabilizer G_{η_t}, with $\eta_t \in M_{\text{reg}}$, fixes pointwise the whole curve η; in particular, all regular orbits $G \cdot \eta_t$ are equivalent to the same homogeneous space G/L;
(4) the decompositions (1.1) associated with the points $\eta_t \in M_{\text{reg}}$ do not depend on t;
(5) there exists a basis $\{f_1, \dots f_n\}$ for $\mathfrak{m}$ such that for any $\eta_t \in M_{\text{reg}}$ the complex structure $J_t : \mathfrak{m} \to \mathfrak{m}$, induced by the complex structure of $T_{\eta_t}M$, is of the following form:

$$J_t f_{2j} = \lambda_j(t) f_{2j+1} \ , \qquad J_t f_{2j+1} = -\frac{1}{\lambda_j(t)} f_{2j} \ ; \tag{1.3}$$

where the function $\lambda_j(t)$ is either one of the functions $-\tanh(t)$, $-\tanh(2t)$, $-\coth(t)$ and $-\coth(2t)$ or it is identically equal to 1.

We call any such curve an *optimal transversal curve*; the basis for $\mathbb{R}Z + \mathfrak{m} \subset \mathfrak{g}$ given by $(Z, f_1, \dots, f_{2n-1})$, where the f_i's verify (1.3), is called *optimal basis associated with* η. An explicit description of the optimal basis for any given semisimple Lie group G is given in §3.

Notice that the family $\mathcal{O}$ of optimal transversal curves depends only on the action of the Lie group G. In particular it is totally independent on the choice of the G-invariant Kähler metric g. At the same time, the Killing fields, associated with the elements of an optimal basis, determine a 1-parameter family of holomorphic frames at the points $\eta_t \in M_{\text{reg}}$, which are orthogonal w.r.t. at least one G-invariant Kähler metric g. It is also proved that, for all K-manifold M which do not belong to a special class of non-standard K-manifold, those holomorphic frames are orthogonal w.r.t. *any* G-invariant Kähler metric g on M (see Corollary 4.2 for details). From these remarks and the fact that $\eta'_t = J\hat{Z}_{\eta_t}$, where Z is the first element of any optimal basis, it may be inferred that any curve $\eta \in \mathcal{O}$ is a reparameterization of a normal geodesics of some (in most cases, *any*) G-invariant Kähler metric on M.

Our second main result is the following. Let η be an optimal transversal curve of a K-manifold, $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ the decomposition (1.1) associated with the regular points $\eta_t \in M_{\text{reg}}$ and let ω and ρ be the Kähler form and the Ricci form, respectively, associated with a given G-invariant Kähler metric g on (M, J).

By a slight modification of arguments used in [PS], we show that there exist two smooth curves

$$Z_\omega, Z_\rho : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a} \ , \qquad \mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathbb{R}Z + \mathfrak{m}) \ , \tag{1.4}$$

satisfying the following properties (here $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l}$ and $C_{\mathfrak{g}}(\mathfrak{l})$ denotes the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$): for any $\eta_t \in M_{\text{reg}}$ and any two element $X, Y \in \mathfrak{g}$, with associated Killing fields $\hat{X}$ and $\hat{Y}$,

$$\omega_{\eta_t}(\hat{X}, \hat{Y}) = \mathcal{B}(Z_\omega(t), [X, Y]) \ , \quad \rho_{\eta_t}(\hat{X}, \hat{Y}) = \mathcal{B}(Z_\rho(t), [X, Y]) \ . \tag{1.5}$$

We call such curves $Z_\omega(t)$ and $Z_\rho(t)$ *the algebraic representatives of* ω *and* ρ *along* η. It is clear that the algebraic representatives determine uniquely the restrictions of ω and ρ to the tangent spaces of the regular orbits. But the following Proposition establishes a result which is somehow stronger.

Before stating the proposition, we recall that in [PS] the following fact was established: if $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ is a decomposition of the form (1.1), then the subalgebra $\mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathbb{R}Z + \mathfrak{m})$ is either 1-dimensional or 3-dimensional and isomorphic with $\mathfrak{su}_2$. By virtue of this dichotomy, the two cases considered in the following proposition are all possible cases.

Proposition 1.1. *Let η_t be an optimal transversal curve of a K-manifold (M, J, g) acted on by the compact semisimple Lie group G and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the decomposition of the form (1.1) determined by the points $\eta_t \in M_{\text{reg}}$. Let also $Z : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$ be the algebraic representative of the Kähler form ω or of the Ricci form ρ. Then:*

(1) *if $\mathfrak{a}$ is 1-dimensional, then it is of the form $\mathfrak{a} = \mathbb{R}Z$ and there exists an element $I \in \mathfrak{z}(\mathfrak{l})$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$ so that*

$$Z(t) = f(t)Z + I \ ; \tag{1.6}$$

(2) *if $\mathfrak{a}$ is 3-dimensional, then it is of the form $\mathfrak{a} = \mathfrak{su}_2 = \mathbb{R}Z + \mathbb{R}X + \mathbb{R}Y$, with $[Z, X] = Y$ and $[X, Y] = Z$ and there exists an element $I \in \mathfrak{z}(\mathfrak{l})$, a real*

number C and a smooth function $f : \mathbb{R} \to \mathbb{R}$ so that

$$Z(t) = f(t) Z_{\mathcal{D}} + \frac{C}{\cosh(t)} X + I \; . \tag{1.7}$$

Conversely, if $Z : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l})$ is a curve in $C_{\mathfrak{g}}(\mathfrak{l})$ of the form (1.6) or (1.7), then there exists a unique closed J-invariant, G-invariant 2-form ϖ on the set of regular points M_{reg}, having $Z(t)$ as algebraic representative.

In particular, the Kähler form ω and the Ricci form ρ are uniquely determined by their algebraic representatives.

Using (1.5), Proposition 1.1 and some basic properties of the decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ (see §5), it can be shown that the algebraic representatives $Z_\omega(t)$ and $Z_\rho(t)$ are uniquely determined by the values $\omega_{\eta_t}(\hat{X}, J\hat{X}) = \mathcal{B}(Z_\omega(t), [X, J_t X])$ and $\rho_{\eta_t}(\hat{X}, J\hat{X}) = \mathcal{B}(Z_\rho(t), [X, J_t X])$, where $X \in \mathfrak{m}$ and J_t is the complex structure on $\mathfrak{m}$ induced by the complex structure of the tangent space $T_{\eta_t} M$.

Here comes our third main result. It consists in Theorem 5.1 and Proposition 5.2, where we give the explicit expression for the value $r_{\eta_t}(X, X) = \rho_{\eta_t}(\hat{X}, J\hat{X})$ for any $X \in \mathfrak{m}$, only in terms of the algebraic representative $Z_\omega(t)$ and of the Lie brackets between X and the elements of the optimal basis in $\mathfrak{g}$. By the previous discussion, this result furnishes a way to write down explicitly the Ricci tensor of the Kähler metric associated with $Z_\omega(t)$.

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Notation. Throughout the paper, if G is a Lie group acting isometrically on a Riemannian manifold M and if $X \in \mathfrak{g} = Lie(G)$, we will adopt the symbol $\hat{X}$ to denote the Killing vector field on M corresponding to X.

The Lie algebra of a Lie group will be always denoted by the corresponding gothic letter. For a group G and a Lie algebra $\mathfrak{g}$, $Z(G)$ and $\mathfrak{z}(\mathfrak{g})$ denote the center of G and of $\mathfrak{g}$, respectively. For any subset A of a group G or of a Lie algebra $\mathfrak{g}$, $C_G(A)$ and $C_{\mathfrak{g}}(A)$ are the centralizer of A in G and $\mathfrak{g}$, respectively.

Finally, for any subspace $\mathfrak{n} \subset \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$, the symbol $\mathfrak{n}^\perp$ denotes the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ w.r.t. the Cartan-Killing form $\mathcal{B}$.

2. Fundamentals of K-manifolds.

2.1 K-manifolds, KO-manifolds and KE-manifolds.

A *K-manifold* is a pair formed by a compact Kähler manifold (M, J, g) and a compact semisimple Lie group G acting almost effectively and isometrically (hence biholomorphically) on M, such that:

i) $b_1(M) = 0$;

ii) G acts of *cohomogeneity one with respect to the action of* G, i.e. the regular G-orbits are of codimension one in M.

In this paper, (M, J, g) will always denote a K-manifold of dimension $2n$, acted on by the compact semisimple Lie group G. We will denote by $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ the Kähler fundamental form and by $\rho = r(\cdot, J\cdot)$ the Ricci form of M.

For the general properties of cohomogeneity one manifolds and of K-manifolds, see e.g. [AA], [AA1], [Br], [HS], [PS]. Here we only recall some properties, which will be used in the paper.

If $p \in M$ is a regular point, let us denote by $L = G_p$ the corresponding isotropy subgroup. Since M is orientable, every regular orbit $G \cdot p$ is orientable. Hence we may consider a unit normal vector field ξ, defined on the subset of regular points M_{reg}, which is orthogonal to any regular orbit. It is known (see [AA1]) that any integral curve of ξ is a geodesic. Any such geodesic is usually called *normal geodesic*.

A normal geodesic γ through a point p verifies the following properties: it intersects any G-orbit orthogonally; the isotropy subalgebra G_{γ_t} at a regular point γ_t is always equal $G_p = L$ (see e.g. [AA], [AA1]). We formalize these two facts in the following definition.

We call *nice transversal curve through a point $p \in M_{\text{reg}}$* any curve $\eta : \mathbb{R} \to M$ with $p \in \eta(\mathbb{R})$ and such that:

i) it intersects any regular orbit;
ii) for any $\eta_t \in M_{\text{reg}}$
$$\eta'_t \notin T_{\eta_t}(G \cdot \eta_t) \; ; \tag{2.1}$$
iii) for any $\eta_t \in M_{\text{reg}}$, $G_{\eta_t} = L = G_p$.

The following property of K-manifold has been proved in [PS].

Proposition 2.1. *Let (M, J, g) be a K-manifold acted on by the compact semisimple Lie group G. Let also $p \in M_{\text{reg}}$ and $L = G_p$ the isotropy subgroup at p. Then:*

(1) *there exists an element Z (determined up to scaling) so that*
$$\mathbb{R}Z \in C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^{\perp} \;, \quad C_{\mathfrak{g}}(\mathfrak{l} + \mathbb{R}Z) = \mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z \; ; \tag{2.2}$$
in particular, the connected subgroup $K \subset G$ with subalgebra $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z$ is the isotropy subgroup of a flag manifold $F = G/K$;

(2) *the dimension of $\mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^{\perp}$ is either 1 or 3; in case $\dim_{\mathbb{R}} \mathfrak{a} = 3$, then $\mathfrak{a}$ is a subalgebra isomorphic to $\mathfrak{su}_2$ and there exists a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ so that $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_{\alpha} + \mathbb{C}E_{\alpha} + \mathbb{C}E_{-\alpha}$ for some root α of the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$.*

Note that if for some regular point p we have that $\dim_{\mathbb{R}} \mathfrak{a} = 1$ (resp. $\dim_{\mathbb{R}} \mathfrak{a} = 3$), then the same occurs at any other regular point. Therefore we may consider the following definition.

Definition 2.2. Let (M, J, g) be a K-manifold and $L = G_p$ the isotropy subgroup of a regular point p. We say that M is a K-manifold *with ordinary action* (or shortly, *KO-manifold*) if $\dim_{\mathbb{R}} \mathfrak{a} = \dim_{\mathbb{R}}(C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^{\perp}) = 1$.

In all other cases, we say that M is *with extra-ordinary action* (or, shortly, *KE-manifold*).

Another useful property of K-manifolds is the following. It can be proved that any K-manifold admits exactly two singular orbits, at least one of which is complex (see [PS1]). By the results in [HS], it also follows that if M is a K-manifold whose singular orbits are both complex, then M admits a G-equivariant blow-up $\tilde{M}$ along the complex singular orbits, which is still a K-manifold and admits a holomorphic fibration over a flag manifold $G/K = G^{\mathbb{C}}/P$, with standard fiber equal to $\mathbb{C}P^1$.

Several other important facts are related to the existence (or non-existence) of two singular complex orbits (see [PS1] for a review of these properties). For this reason, it is convenient to introduce the following definition.

Definition 2.3. We say that a K-manifold M, acted on by a compact semisimple group G with cohomogeneity one, is *standard* if the action of G has two singular complex orbits. We call it *non-standard* in all other cases.

2.2 The CR structure of the regular orbits of a K-manifold.

A *CR structure of codimension* r on a manifold N is a pair $(\mathcal{D}, J)$ formed by a distribution $\mathcal{D} \subset TN$ of codimension r and a smooth family J of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$ on the spaces of the distribution.

A CR structure $(\mathcal{D}, J)$ is called *integrable* if the distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}N$, given by the J-eigenspaces $\mathcal{D}^{10}_x \subset \mathcal{D}^{\mathbb{C}}_x$ corresponding to the eigenvalue $+i$, verifies

$$[\mathcal{D}^{10}, \mathcal{D}^{10}] \subset \mathcal{D}^{10} \ .$$

Note that a complex structure J on manifold N may be always considered as an integrable CR structure of codimension zero.

A smooth map $\phi : N \to N'$ between two CR manifolds $(N, \mathcal{D}, J)$ and $(N', \mathcal{D}', J')$ is called *CR map* (or *holomorphic map*) if:

a) $\phi_*(\mathcal{D}) \subset \mathcal{D}'$;

b) for any $x \in N$, $\phi_* \circ J_x = J'_{\phi(x)} \circ \phi_*|_{\mathcal{D}_x}$.

A *CR transformation* of $(N, \mathcal{D}, J)$ is a diffeomorphism $\phi : N \to N$ which is also a CR map.

Any codimension one submanifold $N \subset M$ of a complex manifold (M, J) is naturally endowed with an integrable CR structure of codimension one $(\mathcal{D}, J)$, which is called *induced CR structure*; it is defined by

$$\mathcal{D}_x = \{ \ v \in T_xN \ : \ Jv \in T_xN \ \} \qquad\qquad J_x = J|_{\mathcal{D}_x} \ .$$

It is clear that any regular orbit $G/L = G \cdot x \in M$ of a K-manifold (M, J, g) has an induced CR structure $(\mathcal{D}, J)$, which is invariant under the transitive action of G. For this reason, several facts on the global structure of the regular orbits of a K-manifolds can be detected using what is known on compact homogeneous CR manifolds (see e.g. [AHR] and [AS]).

Here, we recall some of those facts, which will turn out to be crucial in the next sections.

Let $(G/L, \mathcal{D}, J)$ be a homogeneous CR manifold of a compact semisimple Lie group G, with an integrable CR structure $(\mathcal{D}, J)$ of codimension one. If we consider the $\mathcal{B}$-orthogonal decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{l} = Lie(L)$, then the orthogonal complement $\mathfrak{n}$ is naturally identifiable with the tangent space $T_o(G/L)$, $o = eL$, by means of the map

$$\phi : \mathfrak{n} \to T_o(G/L) \ , \qquad \phi(X) = \hat{X}|_o \ .$$

If we denote by $\mathfrak{m}$ the subspace

$$\mathfrak{m} = \phi^{-1}(\mathcal{D}_o) \subset \mathfrak{n} \ ,$$

we get the following orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{n} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m} \ . \tag{2.3}$$

where $Z_{\mathcal{D}} \in (\mathfrak{l} + \mathfrak{m})^{\perp}$. Since the decomposition is $\mathrm{ad}_{\mathfrak{l}}$-invariant, it follows that $Z_{\mathcal{D}} \in C_{\mathfrak{g}}(\mathfrak{l})$.

Using again the identification map $\phi : \mathfrak{n} \to T_o(G/L)$, we may consider the complex structure

$$J : \mathfrak{m} \to \mathfrak{m} \ , \qquad J \overset{\text{def}}{=} \phi^*(J_o) \ . \tag{2.4}$$

Note that J is uniquely determined by the direct sum decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} \ , \quad \mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}} \ , \tag{2.5}$$

where $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ are the J-eigenspaces with eigenvalues $+i$ and $-i$, respectively.

In all the following, (2.3) will be called *the structural decomposition of* $\mathfrak{g}$ *associated with* $\mathcal{D}$; the subspace $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ (respectively, $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$) given (2.5) will be called *the holomorphic* (resp. *anti-holomorphic*) *subspace associated with* $(\mathcal{D}, J)$.

We recall that a G-invariant CR structure $(\mathcal{D}, J)$ on G/L is integrable if and only if the associated holomorphic subspace $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ is so that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} \quad \text{is a subalgebra of} \quad \mathfrak{g}^{\mathbb{C}} \ . \tag{2.6}$$

We now need to introduce a few concepts which are quite helpful in describing the structure of a generic compact homogeneous CR manifold.

Definition 2.4. Let $N = G/L$ be a homogeneous manifold of a compact semisimple Lie group G and $(\mathcal{D}, J)$ a G-invariant, integrable CR structure of codimension one on N.

We say that a CR manifold $(N = G/L, \mathcal{D}, J)$ is a *Morimoto-Nagano space* if either $G/L = S^{2n-1}$, $n > 1$, endowed with the standard CR structure of $S^{2n-1} \subset \mathbb{C}P^n$, or there exists a subgroup $H \subset G$ so that:

a) G/H is a compact rank one symmetric space (i.e. $\mathbb{R}P^n = \mathrm{SO}_{n+1}/\mathrm{SO}_n \cdot \mathbb{Z}_2$, $S^n = \mathrm{SO}_{n+1}/\mathrm{SO}_n$, $\mathbb{C}P^n = \mathrm{SU}_{n+1}/\mathrm{SU}_n$, $\mathbb{H}P^n = \mathrm{Sp}_{n+1}/\mathrm{Sp}_n$ or $\mathbb{O}P^2 = \mathrm{F}_4/Spin_9$);
b) G/L is a sphere bundle $S(G/H) \subset T(G/H)$ in the tangent space of G/H;
c) $(\mathcal{D}, J)$ is the CR structure induced on $G/L = S(G/H)$ by the G-invariant complex structure of $T(G/H) \cong G^{\mathbb{C}}/H^{\mathbb{C}}$.

If a Morimoto-Nagano space is G-equivalent to a sphere S^{2n-1} we call it *trivial*; we call it *non-trivial* in all other cases.

A G-equivariant holomorphic fibering

$$\pi : N = G/L \to \mathcal{F} = G/Q$$

of $(N, \mathcal{D}, J)$ onto a non-trivial flag manifold $(\mathcal{F} = G/Q, J_{\mathcal{F}})$ with invariant complex structure $J_{\mathcal{F}}$, is called *CRF fibration*. A CRF fibration $\pi : G/L \to G/Q$ is called *nice* if the standard fiber is a non-trivial Morimoto-Nagano space; it is called *very nice* if it is nice and there exists no other nice CRF fibration $\pi' : G/L \to G/Q$ with standard fibers of smaller dimension.

The following Proposition gives necessary and sufficient conditions for the existence of a CRF fibration. The proof can be found in [AS].

Proposition 2.5. *Let G/L be homogeneous CR manifold of a compact semisimple Lie group G, with an integrable, codimension one G-invariant CR structure $(\mathcal{D}, J)$. Let also $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ be the structural decomposition of $\mathfrak{g}$ and $\mathfrak{m}^{10}$ the holomorphic subspace, associated with $(\mathcal{D}, J)$.*

Then G/L admits a non-trivial CRF fibration if and only if there exists a proper parabolic subalgebra $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \subsetneq \mathfrak{g}^{\mathbb{C}}$ (here $\mathfrak{r}$ is a reductive part and $\mathfrak{n}$ the nilradical of $\mathfrak{p}$) such that:

$$a)\ \mathfrak{r} = (\mathfrak{p} \cap \mathfrak{g})^{\mathbb{C}}\ ; \qquad b)\ \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{p}\ ; \qquad c)\ \mathfrak{l}^{\mathbb{C}} \subsetneq \mathfrak{r}\ .$$

In this case, G/L admits a CRF fibration with basis $G/Q = G^{\mathbb{C}}/P$, where Q is the connected subgroup generated by $\mathfrak{q} = \mathfrak{r} \cap \mathfrak{g}$ and P is the parabolic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{p}$.

Let us go back to the regular orbits of a K-manifold (M, J, g) acted on by the compact semisimple group G. We already pointed out that each regular orbit $(G/L = G \cdot x, \mathcal{D}, J)$, endowed with the induced CR structure $(\mathcal{D}, J)$, is a compact homogeneous CR manifold. In the statement of the following Theorem we collect the main results on the one-parameter family of compact homogeneous CR manifolds given by the regular orbits of a K-manifold, which is a direct consequence Th. 3.1 in [PS1] (see also [HS] and [PS] Th.2.4).

Theorem 2.6. *Let (M, J, g) be a K-manifold acted on by the compact semisimple Lie group G.*

(1) *If M is standard, then there exists a flag manifold $(G/K, J_o)$ with a G-invariant complex structure J_o, such that any regular orbit $(G \cdot x = G/L, \mathcal{D}, J)$ of M admits a CRF-fibration $\pi : (G/L, \mathcal{D}, J) \to (G/K, J_o)$ onto $(G/K, J_o)$ with standard fiber S^1.*
(2) *If M is non-standard, then there exists a flag manifold $(G/K, J_o)$ with a G-invariant complex structure J_o such that any regular orbit $(G/L = G \cdot x, \mathcal{D}, J)$ admits a very nice CRF fibration $\pi : (G/L, \mathcal{D}, J) \to (G/K, J_o)$ where the standard fiber K/L is a non-trivial Morimoto-Nagano space of dimension $\dim K/L \geq 3$.*

Furthermore, if the last case occurs, then the fiber K/L of the CRF fibration $\pi : (G/L, \mathcal{D}, J) \to (G/K, J_o)$ has dimension 3 if and only if M is a non-standard KE-manifold and K/L is either $S(\mathbb{R}P^2) \subset T(\mathbb{R}P^2) = \mathbb{C}P^2 \setminus \{\ [z]\ \ : {}^t z \cdot z = 0\}$ or $S(\mathbb{C}P^1) \subset T(\mathbb{C}P^1) = \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \{\ [z] = [w]\ \}$.

3. The optimal transversal curves of a K-manifold.

3.1 Notation and preliminary facts.

If G is a compact semisimple Lie group and $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a given Cartan subalgebra, we will use the following notation:

- $\mathcal{B}$ is the Cartan-Killing form of $\mathfrak{g}$ and for any subspace $A \subset \mathfrak{g}$, $A^{\perp}$ is the $\mathcal{B}$-orthogonal complement to A;
- R is the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$;
- $H_\alpha \in \mathfrak{t}^{\mathbb{C}}$ is the $\mathcal{B}$-dual element to the root α;
- for any $\alpha, \beta \in R$, the scalar product (α, β) is set to be equal to $(\alpha, \beta) = \mathcal{B}(H_\alpha, H_\beta)$;
- E_α is the root vector with root α in the Chevalley normalization; in particular $\mathcal{B}(E_\alpha, E_{-\beta}) = \delta_{\alpha\beta}$, $[E_\alpha, E_{-\alpha}] = H_\alpha$, $[H_\alpha, E_\beta] = (\beta, \alpha) E_\beta$ and $[H_\alpha, E_{-\beta}] = -(\beta, \alpha) E_{-\beta}$;
- for any root α,

$$F_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})\ , \qquad G_\alpha = \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha})\ ;$$

  note that for $\alpha, \beta \in R$

$$\mathcal{B}(F_\alpha, F_\beta) = -\delta_{\alpha\beta} = \mathcal{B}(G_\alpha, G_\beta)\ ,\ \mathcal{B}(F_\alpha, G_\beta) = \mathcal{B}(F_\alpha, H_\beta) = \mathcal{B}(G_\alpha, H_\beta) = 0\ ;$$

- the notation for the roots of a simple Lie algebra is the same of [GOV] and [AS].

Recall that for any two roots α, β, with $\beta \neq -\alpha$, in case $[E_\alpha, E_\beta]$ is non trivial then it is equal to $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ where the coefficients $N_{\alpha,\beta}$ verify the following conditions:

$$N_{\alpha,\beta} = -N_{\beta,\alpha}\ , \quad N_{\alpha,\beta} = -N_{-\alpha,-\beta}\ . \tag{3.1}$$

From (3.1) and the properties of root vectors in the Chevalley normalization, the following well known properties can be derived:

(1) for any $\alpha, \beta \in R$ with $\alpha \neq \beta$

$$[F_\alpha, F_\beta], [G_\alpha, G_\beta] \in \operatorname{span}\{F_\gamma\ , \gamma \in R\}\ , \qquad [F_\alpha, G_\beta] \in \operatorname{span}\{G_\gamma\ , \gamma \in R\}\ ; \tag{3.2}$$

(2) for any $H \in \mathfrak{t}^{\mathbb{C}}$ and any $\alpha, \beta \in R$, $\mathcal{B}(H, [F_\alpha, F_\beta]) = \mathcal{B}(H, [G_\alpha, G_\beta]) = 0$ and

$$\mathcal{B}(H, [F_\alpha, G_\beta]) = i\delta_{\alpha\beta}\mathcal{B}(H, H_\alpha) = \delta_{\alpha\beta}\alpha(iH)\ ; \tag{3.3}$$

Finally, for what concerns the Lie algebra of flag manifolds and of CR manifolds, we adopt the following notation.

Assume that G/K is a flag manifold with invariant complex structure J (for definitions and basic facts, we refer to [Al], [AP], [BFR], [Ni]) and let $\pi : G/L \to G/K$ be a G-equivariant S^1-bundle over G/K. In particular, let us assume that $\mathfrak{l}$ is a codimension one subalgebra of $\mathfrak{k}$. Recall that $\mathfrak{k} = \mathfrak{k}^{ss} + \mathfrak{z}(\mathfrak{k})$, with $\mathfrak{k}^{ss}$ semisimple part of $\mathfrak{k}$. Hence the semisimple part $\mathfrak{l}^{ss}$ of $\mathfrak{l}$ is equal to $\mathfrak{k}^{ss}$ and $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z = (\mathfrak{k}^{ss} + \mathfrak{z}(\mathfrak{k}) \cap \mathfrak{l}) + \mathbb{R}Z$ for some $Z \in \mathfrak{z}(\mathfrak{k})$.

Let $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ be a Cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$ contained in $\mathfrak{k}^{\mathbb{C}}$ and R the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Then we will use the following notation:

- $R_o = \{\alpha \in R \ , \ E_\alpha \in \mathfrak{k} \ \}$;
- $R_{\mathfrak{m}} = \{\alpha \in R \ , \ E_\alpha \in \mathfrak{m} \ \}$;
- for any $\alpha \in R$, we denote by $\mathfrak{g}(\alpha)^{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{ \ E_{\pm\alpha}, H_\alpha\}$ and $\mathfrak{g}(\alpha) = \mathfrak{g}(\alpha)^{\mathbb{C}} \cap \mathfrak{g}$;
- $\mathfrak{m}(\alpha)$ denotes the irreducible $\mathfrak{k}^{\mathbb{C}}$-submoduli of $\mathfrak{m}^{\mathbb{C}}$, with highest weight $\alpha \in R_{\mathfrak{m}}$;
- if $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\beta)$ are equivalent as $\mathfrak{l}^{\mathbb{C}}$-moduli, we denote by $\mathfrak{m}(\alpha) + \lambda\mathfrak{m}(\beta)$ the irreducible $\mathfrak{l}^{\mathbb{C}}$-module with highest weight vector $E_\alpha + \lambda E_\beta$, $\alpha, \beta \in R_{\mathfrak{m}}$, $\lambda \in \mathbb{C}$.

3.2 The structural decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ *determined by the CR structure of a regular orbit.*

The main results of this subsection are given by the following two theorems on the structural decomposition of the regular orbits of a K-manifolds. The first one is a straightforward consequence of definitions, Theorem 2.6 and the results in [PS].

Theorem 3.1. *Let (M, J, g) be a standard K-manifold acted on by the compact semisimple group G and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ and $\mathfrak{m}^{10}$ be the structural decomposition and the holomorphic subspace, respectively, associated with the CR structure $(\mathcal{D}, J)$ of a regular orbit $G/L = G \cdot p$. Let also $J : \mathfrak{m} \to \mathfrak{m}$ be the unique complex structure on $\mathfrak{m}$, which determines the decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \overline{\mathfrak{m}^{10}}$.*

Then, $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}$ is the isotropy subalgebra of a flag manifold K, and the complex structure $J : \mathfrak{m} \to \mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{k}}$-invariant and corresponds to a G-invariant complex structure J on G/K.

In particular, there exists a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ and an ordering of the associated root system R, so that $\mathfrak{m}^{10}$ is generated by the corresponding positive root vectors in $\mathfrak{m}^{\mathbb{C}} = (\mathfrak{k}^{\perp})^{\mathbb{C}}$.

The following theorem describes the structural decomposition and the holomorphic subspace of a regular orbit of a non-standard K-manifold. Also this theorem can be considered as a consequence of Theorem 2.6, but the proof is a little bit more involved.

Theorem 3.2. *Let (M, J, g) be a non-standard K-manifold acted on by the compact semisimple group G and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ and $\mathfrak{m}^{10}$ be the structural decomposition and the holomorphic subspace, respectively, associated with the CR structure $(\mathcal{D}, J)$ of a regular orbit $G/L = G \cdot p$.*

Then there exists a simple subalgebra $\mathfrak{g}_F \subset \mathfrak{g}$ with the following properties:

a) *denote by $\mathfrak{l}_F = \mathfrak{l} \cap \mathfrak{g}_F$, $\mathfrak{l}_o = \mathfrak{l} \cap \mathfrak{g}_F^{\perp}$, $\mathfrak{m}_F = \mathfrak{m} \cap \mathfrak{g}_F$ and $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}_F^{\perp}$; then the pair $(\mathfrak{g}_F, \mathfrak{l}_F)$ is one of those listed in Table 1 and $\mathfrak{g}$ and $\mathfrak{g}_F$ admit the following $\mathcal{B}$-orthogonal decompositions:*

$$\mathfrak{g} = \mathfrak{l}_o + (\mathfrak{l}_F + \mathbb{R}Z_{\mathcal{D}}) + (\mathfrak{m}_F + \mathfrak{m}') , \qquad \mathfrak{g}_F = \mathfrak{l}_F + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}_F \ ;$$

furthermore $[\mathfrak{l}_o, \mathfrak{g}_F] = \{0\}$ and the connected subgroup $K \subset G$ with Lie algebra $\mathfrak{k} = \mathfrak{l}_o + \mathfrak{g}_F$ is the isotropy subalgebra of a flag manifold G/K;

b) *denote by $\mathfrak{m}_F^{10} = \mathfrak{m}_F^{\mathbb{C}} \cap \mathfrak{m}^{10}$; then there exists a Cartan subalgebra $\mathfrak{t}_F^{\mathbb{C}} \subset \mathfrak{l}_F^{\mathbb{C}} + \mathbb{C}Z_{\mathcal{D}}$ and a complex number λ with $0 < |\lambda| < 1$ so that the element $Z_{\mathcal{D}}$, determined up to scaling, and the subspace $\mathfrak{m}_F^{10}$, determined up to an element of the Weyl group and up to complex conjugation, are as listed in Table 1 (see §3.1 for notation):*

$\mathfrak{g}_F$	$\mathfrak{l}_F$	$Z_{\mathcal{D}}$	$\mathfrak{m}_F^{10}$
$\mathfrak{su}_2$	$\{0\}$	$-\frac{i}{2}H_{\varepsilon_1-\varepsilon_2}$	$\mathbb{C}(E_{\varepsilon_1-\varepsilon_2} + \lambda E_{-\varepsilon_1+\varepsilon_2})$
$\mathfrak{su}_{n+1}$	$\mathfrak{su}_{n-2} \oplus \mathbb{R}$	$-\ iH_{\varepsilon_1-\varepsilon_2}$	$(\mathbb{C}(E_{\varepsilon_1-\varepsilon_2}+\lambda^2 E_{-\varepsilon_1+\varepsilon_2})\oplus$ $(\mathfrak{m}(\varepsilon_1-\varepsilon_3)+\lambda\mathfrak{m}(\varepsilon_2-\varepsilon_3))\oplus(\mathfrak{m}(\varepsilon_3-\varepsilon_2)+\lambda\mathfrak{m}(\varepsilon_3-\varepsilon_1))$
$\mathfrak{su}_2 + \mathfrak{su}_2$	$\mathbb{R}$	$-\frac{i}{2}(H_{\varepsilon_1-\varepsilon_2} + H_{\varepsilon'_1-\varepsilon'_2})$	$\mathbb{C}(E_{\varepsilon_1-\varepsilon_2}+\lambda E_{-(\varepsilon'_1-\varepsilon'_2)})\oplus\mathbb{C}(E_{\varepsilon'_1-\varepsilon'_2}+\lambda E_{-(\varepsilon_1-\varepsilon_2)})$
$\mathfrak{so}_7$	$\mathfrak{su}_3$	$-\frac{2i}{3}(H_{\varepsilon_1+\varepsilon_2} + H_{\varepsilon_3})$	$(\mathfrak{m}(\varepsilon_1+\varepsilon_2)+\lambda\mathfrak{m}(-\varepsilon_3))\oplus\overline{\mathfrak{m}(-\varepsilon_3)}+\left(\lambda\overline{\mathfrak{m}(\varepsilon_1+\varepsilon_2)}\right)$
$\mathfrak{f}_4$	$\mathfrak{so}_7$	$-i2H_{\varepsilon_1}$	$(\mathfrak{m}(\varepsilon_1+\varepsilon_2)+\lambda^2\mathfrak{m}(-\varepsilon_1+\varepsilon_2))\oplus$ $(\mathfrak{m}(1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4))+\lambda\mathfrak{m}(1/2(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)))$
$\mathfrak{so}_{2n+1}$	$\mathfrak{so}_{2n-1}$	$-iH_{\varepsilon_1}$	$(\mathfrak{m}(\varepsilon_1+\varepsilon_2)+\lambda\mathfrak{m}(-\varepsilon_1+\varepsilon_2))$
$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n-2}$	$-iH_{\varepsilon_1}$	$(\mathfrak{m}(\varepsilon_1+\varepsilon_2)+\lambda\mathfrak{m}(-\varepsilon_1+\varepsilon_2))$
$\mathfrak{sp}_n$	$\mathfrak{sp}_1+\mathfrak{sp}_{n-2}$	$-iH_{\varepsilon_1+\varepsilon_2}$	$(\mathfrak{m}(2\varepsilon_1)+\lambda^2\mathfrak{m}(-2\varepsilon_2))\oplus(\mathfrak{m}(\varepsilon_1+\varepsilon_3)+\lambda\mathfrak{m}(-\varepsilon_2+\varepsilon_3))$

Table 1

c) *the holomorphic subspace $\mathfrak{m}^{10}$ admits the following orthogonal decomposition*

$$\mathfrak{m}^{10} = \mathfrak{m}_F^{10} + \mathfrak{m}'^{10}$$

where $\mathfrak{m}'^{10} = \mathfrak{m}'^{\mathbb{C}} \cap \mathfrak{m}^{10}$;

d) *the complex structure $J' : \mathfrak{m}' \to \mathfrak{m}'$ associated with the eigenspace decomposition $\mathfrak{m}'^{\mathbb{C}} = \mathfrak{m}'^{10} + \mathfrak{m}'^{01}$, where $\mathfrak{m}'^{01} = \overline{\mathfrak{m}'^{10}}$, is Ad_K-invariant and*

determines a G-invariant complex structure on the flag manifold G/K; in particular the J'-eigenspaces are $\mathrm{ad}_{\mathbb{R}Z_{\mathcal{D}}}$*-invariant:*

$$[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{10}] \subset \mathfrak{m}'^{10} \ , \qquad [\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{01}] \subset \mathfrak{m}'^{01} \ .$$

The proof of Theorem 3.2 needs the following Lemma.

Lemma 3.3. *Let $G/L = G \cdot p$ be a regular orbit of the non-standard K-manifold (M, J, g). Let also $\pi : (G/L, \mathcal{D}, J) \to (G/K, J_o)$ be the CRF fibration given in Theorem 2.6 and $(\mathcal{D}^K, J^K)$ the CR structures of the standard fiber K/L.*
Then:

i) *the 1-dimensional subspaces $\mathbb{R}Z_{\mathcal{D}^K}$ and $\mathbb{R}Z_{\mathcal{D}}$ of the structural decompositions of $\mathfrak{k}$ and $\mathfrak{g}$ at the point p are the same, i.e. their structural decompositions are $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}_K$ and $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + (\mathfrak{m}_K + \mathfrak{m}')$;*
ii) *the holomorphic subspace $\mathfrak{m}^{10}$ of $(G/L, \mathcal{D}, J)$ admits the $\mathcal{B}$-orthogonal decomposition $\mathfrak{m}^{10} = \mathfrak{m}^{10}_K + \mathfrak{m}'^{10}$ where $\mathfrak{m}'^{10} = \mathfrak{m}^{10} \cap \mathfrak{m}'^{\mathbb{C}}$ and $\mathfrak{m}^{10}_K$ is the holomorphic subspace of $(K/L, \mathcal{D}^K, J^K)$;*
iii) *$[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{10}] \subset \mathfrak{m}'^{10}$ and $[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{01}] \subset \mathfrak{m}'^{01}$.*

Proof. Let $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}^K} + \mathfrak{m}_K$ and $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ be the structural decompositions of $\mathfrak{k}$ and $\mathfrak{g}$ at the point p, associated with the CR structures $(\mathcal{D}^K, J^K)$ and $(\mathcal{D}, J)$, respectively. Denote also by J^K and J the induced complex structures on $\mathfrak{m}_K$ and $\mathfrak{m}$.

To prove i), we have to show that $\mathbb{R}Z_{\mathcal{D}^K} = \mathbb{R}Z_{\mathcal{D}}$. This is proved by the following observation. By definitions,

$$\mathfrak{m}_K = \{\ X \in \mathfrak{m} \ : \ \pi_*(\hat{X}_{eL}) = 0\} = \mathfrak{m} \cap \mathfrak{k}$$

and hence

$$\mathbb{R}Z_{\mathcal{D}^K} = \mathfrak{k} \cap (\mathfrak{l} + \mathfrak{m}_K)^{\perp} = \mathfrak{k} \cap (\mathfrak{l} + (\mathfrak{m} \cap \mathfrak{k}))^{\perp} \subseteq \mathfrak{k} \cap (\mathfrak{l} + \mathfrak{m})^{\perp} = \mathfrak{k} \cap \mathbb{R}Z_{\mathcal{D}} = \mathbb{R}Z_{\mathcal{D}} \ .$$

ii) follows from the fact $J_K = J|_{\mathfrak{m}_K}$.

To prove iii), we recall that by Proposition 2.5, if P is the parabolic subgroup such that $(G/K, J_F)$ is G-equivariantly biholomorphic to $G^{\mathbb{C}}/P$, then the subalgebra $\mathfrak{p} = Lie(P) \subset \mathfrak{g}^{\mathbb{C}}$ verifies

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}_K + \mathfrak{m}'^{01} \subset \mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{n}$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{k}^{\mathbb{C}}$ is a reductive complement to $\mathfrak{n}$. In particular, $\mathfrak{m}'^{01} \subset \mathfrak{p} \cap (\mathfrak{k}^{\mathbb{C}})^{\perp} = \mathfrak{n}$. Moreover,

$$\dim_{\mathbb{C}} \mathfrak{m}'^{01} = \dim_{\mathbb{C}} G/P = \dim_{\mathbb{C}} \mathfrak{n}$$

and hence $\mathfrak{m}'^{01} = \mathfrak{n}$. It follows that $[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{01}] \subset [\mathfrak{k}^{\mathbb{C}}, \mathfrak{n}] \subset \mathfrak{n} = \mathfrak{m}'^{01}$ and $[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{10}] = \overline{[\mathbb{R}Z_{\mathcal{D}}, \mathfrak{m}'^{01}]} \subset \overline{\mathfrak{m}'^{01}} = \mathfrak{m}'^{10}$. $\square$

Proof of Theorem 3.2. Let $K \subset G$ be a subgroup so that any regular orbit G/L admits a very nice CRF fibration $\pi : (G/L, \mathcal{D}, J) \to (G/K, J_o)$ as prescribed by Theorem 2.6. Then, for any regular point p, the K-orbit $K/L = K \cdot p \subset G/L = G \cdot p$ (which is the fiber of the CRF fibration π) is a non-trivial Morimoto-Nagano space. In particular, K/L is Levi non-degenerate, it is simply connected and the CR structure is non-standard (for the definition of non-standard CR structures and the properties of the CR structures of the Morimoto-Nagano spaces, see [AS]). Furthermore, by Lemma 3.3, the 1-dimensional subspace $\mathbb{R}Z_{\mathcal{D}^K}$ associated with the CR structure of K/L coincides with the 1-dimensional subspace $\mathbb{R}Z_{\mathcal{D}}$ associated with the CR structure of G/L.

Let $L_o \subset L$ be the normal subgroup of the elements which act trivially on K/L. Let also $G_F = K/L_o$ and $\mathfrak{l}_o = Lie(L_o)$, $\mathfrak{g}_F = \mathfrak{k} \cap (\mathfrak{l}_o)^\perp \cong Lie(G_F)$.

Note that Th. 1.3, 1.4 and 1.5 of [AS] apply immediately to the homogeneous CR manifold G_F/L_F, with $L_F = L \mod L_o$. In particular, since the CRF fibration $\pi : G/L \to G/K$ is very nice, $K/L = G_F/L_F$ is a primitive homogeneous CR manifold (for the definition of primitive CR manifolds, see [AS]) and $\mathfrak{g}_F$ is $\mathfrak{su}_n$, $\mathfrak{su}_2 + \mathfrak{su}_2$, $\mathfrak{so}_7$, $\mathfrak{f}_4$, $\mathfrak{so}_n$ $(n \geq 5)$ or $\mathfrak{sp}_n$ $(n \geq 2)$.

From Th.1.4, Prop. 6.3 and Prop. 6.4 in [AS] and from Lemma 3.3 i) and ii), it follows immediately that the subalgebra $\mathfrak{g}_F$ and the holomorphic subspace $\mathfrak{m}^{10}_F$, associated with the CR structure of the fiber $K/L = G_F/L_F$, verify a), b), c) and d). $\square$

In the following, we will call the subalgebra $\mathfrak{g}_F$ *the Morimoto-Nagano subalgebra of the non-standard K-manifold M*. We will soon prove that the Morimoto-Nagano subalgebra is independent (up to conjugation) from the choice of the regular orbit $G \cdot p = G/L$.

We will also call $(\mathfrak{g}_F, \mathfrak{l}_F)$ and the subspace $\mathfrak{m}^{10}_F$ the *Morimoto-Nagano pair* and the *Morimoto-Nagano holomorphic subspace*, respectively, of the regular orbit $G/L = G \cdot p$.

3.3 Optimal transversal curves.

We prove now the existence of a special family of nice transversal curves called optimal transversal curves (see §1). We first show the existence of such curves for a non-standard K-manifold.

Theorem 3.4. *Let (M, J, g) be a non-standard K-manifold acted on by the compact semisimple group G. Then there exists a point p_o in the non-complex singular orbit and an element $Z \in \mathfrak{g}$, such that the curve*

$$\eta : \mathbb{R} \to M \ , \qquad \eta_t = \exp(tiZ) \cdot p_o$$

verifies the following properties:

(1) *it is a nice transversal curve; in particular the isotropy subalgebra $\mathfrak{g}_{\eta_t}$ for any $\eta_t \in M_{\text{reg}}$ is a fixed subalgebra $\mathfrak{l}$;*

(2) *there exists a subspace* $\mathfrak{m}$ *such that, for any* $\eta_t \in M_{\mathrm{reg}}$*, the structural decomposition* $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}(t) + \mathfrak{m}(t)$ *of the orbit* $G/L = G \cdot \eta_t$ *is given by* $\mathfrak{m}(t) = \mathfrak{m}$ *and* $\mathbb{R}Z_{\mathcal{D}}(t) = \mathbb{R}Z$*;*

(3) *the Morimoto-Nagano pairs* $(\mathfrak{g}_F(t), \mathfrak{l}_F(t))$ *of the regular orbits* $G \cdot \eta_t$ *do not dependent on* t*;*

(4) *for any* $\eta_t \in M_{\mathrm{reg}}$*, the holomorphic subspace* $\mathfrak{m}^{10}(t)$ *admits the orthogonal decomposition*

$$\mathfrak{m}^{10}(t) = \mathfrak{m}^{10}_F(t) + \mathfrak{m}'^{10}(t)$$

where $\mathfrak{m}'^{10}(t) = \mathfrak{m}'^{10} \subset \mathfrak{m}^{\mathbb{C}}$ *is independent on* t *and* $\mathfrak{m}^{10}_F(t)$ *is a Morimoto-Nagano holomorphic subspace which is listed in Table 1, determined by the parameter* λ *equal to*

$$\lambda = \lambda(t) = e^{2t} \ .$$

Moreover, if $\eta_t = \exp(tiZ) \cdot p_o$ *is any of such curves and if* $(\mathfrak{g}_F, \mathfrak{l}_F)$ *is (up to conjugation) the Morimoto-Nagano pair of a regular orbits* $G/L = G \cdot \eta_t$*, then (up to conjugation)* Z *is the element in the column "*$Z_{\mathcal{D}}$*" of Table 1, associated with the Lie algebra* $\mathfrak{g}_F$*.*

For the proof of Theorem 3.4, we first need two Lemmata.

Lemma 3.5. *Let* (M, J, g) *be a K-manifold acted on by the compact semisimple Lie group* G*. Let also* p *be a regular point and* $G/L = G \cdot p$ *and* $G^{\mathbb{C}}/H = G^{\mathbb{C}} \cdot p$ *the* G*- and the* $G^{\mathbb{C}}$*-orbit of* p*, respectively. Then:*

(1) *the isotropy subalgebra* $\mathfrak{h} = Lie(G^{\mathbb{C}}_p)$ *is equal to*

$$\mathfrak{h} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$$

where $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$ *is the anti-holomorphic subspace associated with the CR structure of* $G/L = G \cdot p$*;*

(2) *for any* $g \in G^{\mathbb{C}}$*, the isotropy subalgebra* $\mathfrak{l}' = \mathfrak{g}_{p'}$ *at* $p' = g \cdot p$ *is equal to*

$$\mathfrak{l}' = \mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \cap \mathfrak{g} \ ;$$

(3) *let* $g \in G^{\mathbb{C}}$ *and suppose that* $p' = g \cdot p$ *is a regular point; if we denote by* $\mathfrak{g} = \mathfrak{l}' + \mathbb{R}Z'_{\mathcal{D}} + \mathfrak{m}'$ *and by* $\mathfrak{m}'^{10}$ *the structural decomposition and the holomorphic subspace, respectively, given by the CR structure of* $G \cdot p' = G/L'$*, then*

$$\mathfrak{m}'^{10} = \overline{\mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \overline{\mathfrak{m}^{10}})} \ ,$$

$$\mathfrak{m}' = \left(\mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \overline{\mathfrak{m}^{10}}) + \overline{\mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \overline{\mathfrak{m}^{10}})}\right) \cap \mathfrak{g} \cap \mathfrak{l}'^{\perp} \ .$$

Proof. (1) Consider an element $V = X + iY \in \mathfrak{g}^{\mathbb{C}}$, with $X, Y \in \mathfrak{g}$. Then V belongs to $\mathfrak{h}$ if and only if

$$\widehat{X + iY}|_p = \hat{X}_p + J\hat{Y}_p = 0 \ .$$

This means that $J\hat{X}_p = -\hat{Y}_p$ is tangent to the orbit $G\cdot p$. In particular, $X, Y \in \mathfrak{l}+\mathfrak{m}$ and $V = X + iJX \in \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$.

(2) Clearly, $L' = G \cap G^{\mathbb{C}}_{p'} = G \cap (gHg^{-1})$ and $\mathfrak{l}' = \mathfrak{g} \cap \mathrm{Ad}_g(\mathfrak{h})$. The claim is then an immediate consequence of (1).

(3) From (1), it follows that

$$\mathfrak{m}'^{10} = \overline{\mathfrak{m}'^{01}} = \overline{\mathfrak{h}' \cap (\mathfrak{l}'^{\mathbb{C}})^{\perp}} = \overline{\mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})} \cap (\mathfrak{l}'^{\mathbb{C}})^{\perp} \ .$$

From this, the conclusion follows. □

Lemma 3.6. *Let (M, J, g) be a K-manifold acted on by the compact semisimple Lie group G. Let also p be a regular point and $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ the structural decomposition associated with the CR structure of $G/L = G\cdot p$. Then:*

1. *for any $g \in \exp(\mathbb{C}^* Z_{\mathcal{D}})$, the isotropy subalgebra $\mathfrak{g}_{p'}$ at the point $p' = g\cdot p$ is orthogonal to $\mathbb{R}Z_{\mathcal{D}}$; moreover, $\mathfrak{l} \subseteq \mathfrak{g}_{p'}$ and, if p' is regular, $\mathfrak{l} = \mathfrak{g}_{p'}$;*
2. *the curve*
$$\eta : \mathbb{R} \to M \ , \qquad \eta_t = \exp(itZ_{\mathcal{D}})\cdot p$$
is a nice transversal curve through p.

Proof. (1) From Lemma 3.5 (2), for any point $p' = \exp(\lambda Z_{\mathcal{D}})\cdot p$, with $\lambda \in \mathbb{C}^*$,

$$\mathcal{B}(\mathfrak{g}_{p'}, \mathbb{R}Z_{\mathcal{D}}) = \mathcal{B}(\mathrm{Ad}_{\exp(\lambda Z_{\mathcal{D}})}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \cap \mathfrak{g}, \mathbb{R}Z_{\mathcal{D}}) =$$

$$= \mathcal{B}((\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \cap \mathfrak{g}, \mathrm{Ad}_{\exp(-\lambda Z_{\mathcal{D}})}(\mathbb{R}Z_{\mathcal{D}})) = \mathcal{B}((\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \cap \mathfrak{g}, \mathbb{R}Z_{\mathcal{D}}) = 0 \ .$$

Moreover, since $Z_{\mathcal{D}} \in C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{l}^{\mathbb{C}})$, we get that

$$\mathfrak{g}_{p'} = (\mathrm{Ad}_{\exp(\lambda Z_{\mathcal{D}})}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})) \cap \mathfrak{g} = \mathfrak{l} + \mathrm{Ad}_{\exp(\lambda Z_{\mathcal{D}})}(\mathfrak{m}^{01}) \cap \mathfrak{g} \supset \mathfrak{l} \ .$$

This implies that $\mathfrak{l} = \mathfrak{g}_{p'}$ if p' is regular.

(2) From (1), we have that condition (2.1) and the equality $G\cdot\eta_t = G\cdot p = G/L$ are verified for any point $\eta_t \in M_{\mathrm{reg}}$. It remains to show that η intersects any regular orbit.

Let $\Omega = M\backslash G$ be the orbit space and $\pi : M \to \Omega = M\backslash G$ the natural projection map. It is known (see e.g. [Br]) that Ω is homeomorphic to $\Omega = [0, 1]$, with $M_{\mathrm{reg}} = \pi^{-1}(]0, 1[)$. Hence η intersects any regular orbit if and only if $(\pi\circ\eta)(\mathbb{R}) \supset]0, 1[$.

Let $x_1 = \inf(\pi\circ\eta)(\mathbb{R})$ and let $\{t_n\} \subset]0, 1[$ be a sequence such that $(\pi\circ\eta)_{t_n}$ tends to x_1. If we assume that $x_1 > 0$, we may select a subsequence t_{n_k} so that $\lim_{n_k\to\infty} \eta_{t_{n_k}}$ exists and it is equal to a regular point p_o. From (1) and a continuity argument, we could conclude that $\mathfrak{l}$ is equal to the isotropy subalgebra $\mathfrak{g}_{p_o}$, that $\hat{Z}_{\mathcal{D}}|_{p_o} \neq 0$ and that $J\hat{Z}_{\mathcal{D}}|_{p_o}$ is not tangent to the orbit $G\cdot p_o$. In particular, it would follow that the curve $\exp(i\mathbb{R}Z_{\mathcal{D}})\cdot p_o$ has non-empty intersection with $\eta(\mathbb{R}) = \exp(i\mathbb{R}Z_{\mathcal{D}})\cdot p$ and that $p_o \in \eta(\mathbb{R})$; moreover we would have that η is transversal to $G\cdot p_o$ and that $x_1 = \pi(p_o)$ is an inner point of $\pi\circ\eta(\mathbb{R})$, which is a contradiction.

A similar contradiction arises if we assume that $x_2 = \sup\pi\circ\eta(\mathbb{R}) < 1$. □

Proof of Theorem 3.4. Pick a regular point p. Let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$ be the structural decomposition of the orbit $G \cdot p$ and let $\eta_t = \exp(itZ_{\mathcal{D}}) \cdot p$. From Lemmata 3.5 and 3.6 and Theorem 3.2, the structural decompositions $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}(t) + \mathfrak{m}(t)$ of all regular orbits $G \cdot \eta_t$ are independent on t. Moreover, from Lemma 3.5 and Theorem 3.2, it follows that the Morimoto-Nagano pair $(\mathfrak{g}_F, \mathfrak{l}_F)$ is the same for all regular orbits $G \cdot \eta_t$ and the holomorphic subspace $\mathfrak{m}^{10}_t$ of the orbit $G \cdot \eta_t$ is of the form

$$\mathfrak{m}^{10}_t = \overline{\mathrm{Ad}_{\exp(itZ_{\mathcal{D}})}(\overline{\mathfrak{m}^{10}_0})} = \mathrm{Ad}_{\exp(-itZ_{\mathcal{D}})}(\mathfrak{m}^{10}_F(0)) + \mathrm{Ad}_{\exp(-itZ_{\mathcal{D}})}(\mathfrak{m}'^{10}(0)) \tag{3.4}$$

where $\mathfrak{m}^{10}_0 = \mathfrak{m}^{10}_F(0) + \mathfrak{m}'^{10}(0)$ is the decomposition of the holomorphic subspace of $G \cdot \eta_0$ given in Theorem 3.2 c). Since $Z_{\mathcal{D}} \in \mathfrak{g}_F$, from (3.4) and Theorem 3.2 d), it follows that

$$\mathfrak{m}^{10}_t = \mathrm{Ad}_{\exp(-itZ_{\mathcal{D}})}(\mathfrak{m}^{10}_F(0)) + \mathfrak{m}'^{10}(0) \ .$$

This proves that the Morimoto-Nagano holomorphic subspace $\mathfrak{m}^{10}_F(t)$ of the orbit $G \cdot \eta_t$ is

$$\mathfrak{m}^{10}_F(t) = \mathrm{Ad}_{\exp(-itZ_{\mathcal{D}})}(\mathfrak{m}^{10}_F(0)) \tag{3.5}$$

and that the $\mathcal{B}$-orthogonal complement $\mathfrak{m}'^{10} = \mathfrak{m}'^{10}(0)$ is independent on t and $\mathrm{ad}_{Z_{\mathcal{D}}}$-invariant.

A simple computation shows that if $\mathfrak{g}_F$ and $\mathfrak{m}^{10}_F(t) = \mathrm{Ad}_{\exp(-itZ_{\mathcal{D}})}(\mathfrak{m}^{10}_F(0))$ appear in a row of Table 1 and if $Z_{\mathcal{D}}$ is equal to $Z_{\mathcal{D}} = AZ_o$, where Z_o is the corresponding element listed in the column "$Z_{\mathcal{D}}$", then $\mathfrak{m}^{10}_F(t)$ is determined by a complex parameter $\lambda = \lambda(t)$, which verifies the differential equation

$$\frac{d\lambda}{dt} = 2A\lambda(t) \ .$$

In particular, if we assume $A = 1$, then $\lambda(t) = e^{2t+B_p}$ where B_p is a complex number which depends only on the regular point p.

Let us replace p with the point $p_o = \exp(-i\frac{B_p}{2}Z) \cdot p$: it is immediate to realize that the new function $\lambda(t)$ is equal to

$$\lambda(t) = e^{2t+B_p-B_p} = e^{2t} \ .$$

This proves that the curve $\eta_t = e^{itZ_{\mathcal{D}}} \cdot p_o$ verifies (1), (2), (3) and (4).

It remains to prove that for any choice of the regular point p, the point $p_o = \exp(-i\frac{B_p}{2}Z) \cdot p$ is a point of the non-complex singular orbit of M.

Observe that, since $\eta(\mathbb{R})$ is the orbit of a real 1-parameter subgroup of $G^{\mathbb{C}}$, the complex isotropy subalgebra $\mathfrak{h}_t \subset \mathfrak{g}^{\mathbb{C}}$ is (up to conjugation) independent on the point η_t. Indeed, if η_{t_o} is a regular point with complex isotropy subalgebra $\mathfrak{h}_{t_o} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}_F + \mathfrak{m}'^{01}$, then for any other point η_t, we have that

$$\mathfrak{h}_t = \mathrm{Ad}_{\exp(i(t-t_o)Z_{\mathcal{D}})}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}_F + \mathfrak{m}'^{01}) \ .$$

On the other hand, the real isotropy subalgebra $\mathfrak{g}_{\eta_t} \subset \mathfrak{g}$ is equal to

$$\mathfrak{g}_{\eta_t} = \mathfrak{h}_t \cap \mathfrak{g} = \mathrm{Ad}_{\exp(i(t-t_o)Z_{\mathcal{D}})}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}_F + \mathfrak{m}'^{01}) \cap \mathfrak{g} \ . \tag{3.6}$$

From (3.6), Table 1 and (4), one can check that in all cases

$$\mathfrak{g}_{\eta_0} \supsetneq \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}$$

and hence that $\eta_0 = p_o$ is a singular point for the G-action. On the other hand p_o cannot be in the complex singular G-orbit, because otherwise this orbit would coincide with $G^{\mathbb{C}} \cdot p_o = G^{\mathbb{C}} \cdot p$ and it would contradict the assumption that p is a regular point for the G-action. $\square$

The following is the analogous result for standard K-manifolds.

Theorem 3.7. *Let (M, J, g) be a standard K-manifold acted on by the compact semisimple group G and let p_o be any regular point for the G-action. Let also $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ and $\mathfrak{m}^{10}$ be the structural decomposition and the holomorphic subspace associated with the CR structure of the orbit $G/L = G \cdot p_o$. Then the curve*

$$\eta : \mathbb{R} \to M \ , \qquad \eta_t = \exp(tiZ) \cdot p_o$$

verifies the following properties:

1. *it is a nice transversal curve; in particular the stabilizer in $\mathfrak{g}$ of any regular point η_t is equal to the isotropy subalgebra $\mathfrak{l} = \mathfrak{g}_{p_o}$;*
2. *for any regular point η_t, the structural decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}(t) + \mathfrak{m}(t)$ and the holomorphic subspace $\mathfrak{m}^{10}(t)$ of the CR structure of $G/L = G \cdot \eta_t$ is given by the subspaces $\mathfrak{m}(t) = \mathfrak{m}$, $\mathbb{R}Z_{\mathcal{D}}(t) = \mathbb{R}Z$ and $\mathfrak{m}^{10}(t) = \mathfrak{m}^{10}$.*

Proof. (1) is immediate from Lemma 3.6.

(2) It is sufficient to prove that $[Z, \mathfrak{m}^{10}] \subset \mathfrak{m}^{10}$. In fact, from this the claim follows as an immediate corollary of Lemmata 3.5 and 3.6.

Let $(G/K, J_F)$ be the flag manifold with invariant complex structure J_F, given by Theorem 2.6, so that any regular orbit $G \cdot x$ admits a CRF fibration onto G/K, with fiber S^1. Let also P be the parabolic subalgebra of $G^{\mathbb{C}}$ such that G/K is biholomorphic to $G^{\mathbb{C}}/P$.

From Proposition 2.5, if we denote by $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{n}$ the decomposition of the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ into nilradical $\mathfrak{n}$ plus reductive part $\mathfrak{k}^{\mathbb{C}}$, we have that

$$\mathfrak{k} = \mathfrak{p} \cap \mathfrak{g} \ , \qquad \mathfrak{l}^{\mathbb{C}} \subsetneq \mathfrak{k}^{\mathbb{C}} \ , \qquad \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{k}^{\mathbb{C}} + \mathfrak{n} \ . \tag{3.7}$$

Since the CRF fibration has fiber S^1, it follows that $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z'$ for some $Z' \in \mathfrak{z}(\mathfrak{k}) \subset \mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^{\perp}$.

In case $\dim \mathfrak{a} = 1$, we have that $\mathfrak{a} = \mathbb{R}Z = \mathbb{R}Z'$ and hence $\mathfrak{m}^{10} \subset (\mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z)^{\perp} = (\mathfrak{k}^{\mathbb{C}})^{\perp}$. From (3.7) we get that $\mathfrak{m}^{01} = \mathfrak{n}$ and that $[Z, \mathfrak{m}^{01}] \subset [\mathfrak{k}^{\mathbb{C}}, \mathfrak{n}] \subset \mathfrak{n} = \mathfrak{m}^{01}$.

In case $\mathfrak{a}$ is 3-dimensional, let us denote by $\mathfrak{a}^{\perp} = \mathfrak{a} \cap \mathfrak{m} = \mathfrak{a} \cap (\mathbb{R}Z)^{\perp}$ and by $\mathfrak{a}^{10} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{10}$, $\mathfrak{a}^{01} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{01} = \overline{\mathfrak{a}^{10}}$ so that $(\mathfrak{a}^{\perp})^{\mathbb{C}} = \mathfrak{a}^{10} + \mathfrak{a}^{01}$. Consider also the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{a}^{\perp} + \mathfrak{m}' \ , \qquad \mathfrak{m}^{10} = \mathfrak{a}^{10} + \mathfrak{m}'^{10} \ ,$$

where $\mathfrak{m}'^{10} = \mathfrak{m}^{10} \cap \mathfrak{m}'^{\mathbb{C}}$. Let $\mathfrak{l}^{ss}$ be the semisimple part of $\mathfrak{l}$ and note that $\mathfrak{l}^{ss} = \mathfrak{k}^{ss}$. By classical properties of flag manifolds (see e.g. [Al], [AP], [Ni]) the $\mathrm{ad}_{\mathfrak{k}^{ss}}$-module $\mathfrak{m}'$ contains no trivial $\mathrm{ad}_{\mathfrak{k}^{ss}}$-module and hence $\mathfrak{m}'^{10} = [\mathfrak{k}^{ss}, \mathfrak{m}'^{10}] = [\mathfrak{k}, \mathfrak{m}'^{10}]$. In particular, $\mathfrak{m}'^{01} = \overline{\mathfrak{m}'^{10}}$ is orthogonal to $\mathfrak{k}^{\mathbb{C}}$ and hence it is included in $\mathfrak{n}$. So,

$$[Z, \mathfrak{m}'^{01}] \subset [Z, \mathfrak{n} \cap (\mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}})^{\perp}] \subset \mathfrak{n} \cap (\mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}})^{\perp} = \mathfrak{m}'^{01} .$$

From this, it follows that in order to prove that $[Z, \mathfrak{m}^{10}] \subset \mathfrak{m}^{10}$, one has only to show that $[Z, \mathfrak{a}^{10}] \subset \mathfrak{a}^{10} \subset \mathfrak{m}^{10}$.

By dimension counting, $\mathfrak{a}^{10} = \mathbb{C}E$ for some element $E \in \mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$. In case E is a nilpotent element for the Lie algebra $\mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$, we may choose a Cartan subalgebra $\mathbb{C}H_\alpha$ for $\mathfrak{a} = \mathfrak{sl}_2(\mathbb{R})$, so that $E \in \mathbb{C}E_\alpha$. In this case, we have that

$$Z \in (\mathfrak{a}^{10} + \mathfrak{a}^{01})^{\perp} = (\mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha})^{\perp} = \mathbb{C}H_\alpha$$

and hence $[Z, \mathfrak{a}^{10}] \subset [\mathbb{C}H_\alpha, \mathbb{C}E_\alpha] = \mathbb{C}E_\alpha = \mathfrak{a}^{10}$ and we are done.

In case E is a regular element for $\mathfrak{a}^{\mathbb{C}}$, with no loss of generality, we may consider a Cartan subalgebra $\mathbb{C}H_\alpha$ for $\mathfrak{a}^{\mathbb{C}}$ so that $\mathbb{C}E = \mathbb{C}(E_\alpha + tE_{-\alpha})$ for some $t \neq 0$. In this case, $\mathfrak{a}^{01} = \overline{\mathfrak{a}^{10}} = \mathbb{C}(E_{-\alpha} + \bar{t}E_\alpha) = \mathbb{C}(E_\alpha + \frac{1}{\bar{t}}E_{-\alpha})$ and, since $\mathfrak{a}^{10} \cap \mathfrak{a}^{01} = \{0\}$, it follows that $t \neq 1/\bar{t}$. In particular, we get that $\mathbb{C}Z = (\mathfrak{a}^{10} + \mathfrak{a}^{01})^{\perp} = \mathbb{C}H_\alpha$. Now, by Lemma 3.5 (2), for any $\lambda \in \mathbb{C}^*$, the isotropy subalgebra $\mathfrak{l}_{g_\lambda \cdot p_o}$, with $g_\lambda = \exp(\lambda Z)$, is equal to

$$\mathfrak{l}_{g_\lambda \cdot p_o} = Ad_{\exp(\lambda Z)}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{01} + \mathfrak{m}'^{01}) \cap \mathfrak{g} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}'^{01} + \mathbb{C}(E_\alpha + te^{-2\lambda\alpha(Z)}E_{-\alpha}) \cap \mathfrak{g} .$$

Therefore, if λ is such that $te^{-2\lambda\alpha(Z)} = -1$, we have that $\mathfrak{l}_{g_\lambda \cdot p_o} = \mathfrak{l} + \mathbb{R}(E_\alpha - E_{-\alpha}) \supsetneq \mathfrak{l}$ and hence that $p = g_\lambda \cdot p_o$ is a singular point for the G-action. On the other hand, p is in the $G^{\mathbb{C}}$-orbit of p_o and hence the singular orbit $G \cdot p$ is not a complex orbit. But this is in contradiction with the hypothesis that M is standard and hence that it has two singular G-orbits, which are both complex. □

Any curve $\eta_t = \exp(itZ) \cdot p_o$, which verifies the claim of Theorems 3.4 or 3.7, will be called *optimal transversal curve*.

3.4 The optimal bases along the optimal transversal curves.

In all the following, η is an optimal transversal curve. In case M is a non-standard K-manifold, we denote by $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$, $(\mathfrak{g}_F, \mathfrak{l}_F)$, $\mathfrak{m}^{10}_F(t)$ and $\mathfrak{m}^{10} = \mathfrak{m}^{10}_F(t) + \mathfrak{m}'^{10}$ the structural decomposition, the Morimoto-Nagano pair, the Morimoto-Nagano subspace and the holomorphic subspace, respectively, at the regular points $\eta_t \in M_{\text{reg}}$. The same notation will be adopted in case M is a standard K-manifold, with the convention that, in this case, the Morimoto-nagano pair $(\mathfrak{g}_F, \mathfrak{l}_F)$ is the trivial pair $(\{0\}, \{0\})$ and that the Morimoto-Nagano holomorphic subspace is $\mathfrak{m}^{10}_F = \{0\}$.

We will also assume that $\mathfrak{l} = \mathfrak{l}_o + \mathfrak{l}_F$, where $\mathfrak{l}_o = \mathfrak{l} \cap \mathfrak{l}_F^{\perp}$. By $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}}_o + \mathfrak{t}^{\mathbb{C}}_F \subset \mathfrak{l}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, with $\mathfrak{t}_o \subset \mathfrak{l}_o$ and $\mathfrak{t}_F \subset \mathfrak{l}_F$, we denote a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with the property

that, the expressions of $\mathfrak{m}^{10}_F(t)$ and $Z_{\mathcal{D}}$ in terms of the root vectors of $(\mathfrak{g}^{\mathbb{C}}_F, \mathfrak{t}^{\mathbb{C}}_F)$ are exactly as those listed in Table 1, corresponding to the parameter $\lambda_t = e^{2t}$.

Let R be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Then R is union of the following disjoint subsets of roots:

$$R = R^o \cup R' = (R^o_\perp \cup R^o_F) \cup (R'_F \cup R'_+ \cup R'_-) \ ,$$

where

$$R^o_\perp = \{ \ \alpha \ , \ E_\alpha \in \mathfrak{l}^{\mathbb{C}}_o \ \} \quad , \quad R^o_F = \{ \ \alpha \ , \ E_\alpha \in \mathfrak{l}^{\mathbb{C}}_F \ \} \ ,$$

$$R'_F = \{ \ \alpha \ , \ E_\alpha \in \mathfrak{m}^{\mathbb{C}}_F \ \} \quad , \quad R'_+ = \{ \ \alpha \ , \ E_\alpha \in \mathfrak{m}'^{10} \ \} \quad , \quad R'_- = \{ \ \alpha \ , \ E_\alpha \in \mathfrak{m}'^{01} \ \} \ .$$

Note that

$$-R^o_\perp = R^o_\perp \ , \quad -R^o_F = R^o_F \ , \quad -R'_F = R'_F \ , \quad -R'_+ = R'_- \ .$$

Moreover, $R^o_\perp$ is orthogonal to R^o_F and $R^o_\perp$, R^o_F and $R^o_F \cup R'_F$ are closed subsystems.

Clearly, in case M is standard, we will assume that $R^o_F = R'_F = \emptyset$.

We claim that for any $\alpha \in R'_F$ there exists exactly one root $\alpha^d \in R'_F$ and two integers $\epsilon_\alpha = \pm 1$ and $\ell_\alpha = \pm 1, \pm 2$ such that, for any $t \in \mathbb{R}$,

$$E_\alpha + e^{2\ell_\alpha t} \epsilon_\alpha E_{-\alpha^d} \in \mathfrak{m}^{10}_F(t) \ , \ . \tag{3.8}$$

The proof of this claim is the following. By direct inspection of Table 1, the reader can check that any maximal $\mathfrak{l}^{\mathbb{C}}_F$-isotopic subspace of $\mathfrak{m}^{\mathbb{C}}_F(t)$ (i.e. any maximal subspace which is sum of equivalent irreducible $\mathfrak{l}^{\mathbb{C}}_F$-moduli) is direct sum of exactly two irreducible $\mathfrak{l}^{\mathbb{C}}_F$-moduli (see also [AS]). Let us denote by $(\alpha_i, -\alpha^d_i)$ $(i = 1, 2, \dots)$ all pairs of roots in R_F with the property that the associated root vectors E_{α_i} and $E_{-\alpha^d_i}$ are maximal weight vectors of equivalent $\mathfrak{l}^{\mathbb{C}}_F$-moduli in $\mathfrak{m}^{\mathbb{C}}_F(t)$. Using Table 1, one can check that in all cases $\mathfrak{m}^{10}_F(t)$ decomposes into non-equivalent irreducible $\mathfrak{l}^{\mathbb{C}}_F$-moduli, with maximal weight vectors of the form

$$E_{\alpha_i} + \lambda^{(i)}_t E_{-\alpha^d_i}$$

where $\lambda^{(i)}_t = (\lambda(t))^{\ell_i} = e^{t\ell_i t}$, where ℓ_i is an integer which is either ± 1 or ± 2.

Hence $\mathfrak{m}^{10}_F(t)$ is spanned by the vectors $E_{\alpha_i} + \lambda_t E_{-\alpha^d_i}$ and by vectors of the form

$$[E_\beta, E_{\alpha_i} + \lambda_t E_{-\alpha^d_i}] = N_{\beta,\alpha_i} E_{\alpha_i + \beta} + \lambda_t N_{\beta, -\alpha^d_i} E_{-\alpha^d_i + \beta} \ , \tag{3.9}$$

for some $E_\beta \in \mathfrak{l}^{\mathbb{C}}$. Since the $\mathfrak{l}^{\mathbb{C}}$-moduli containing E_{α_i} and $E_{-\alpha^d_i}$ are equivalent, the lengths of the sequences of roots $\alpha_i + r\beta$ and $-\alpha^d_i + r\beta$ are both equal to some given integer, say p. This implies that for any root $\beta \in R^o_F$

$$N^2_{\beta,\alpha_i} = (p+1)^2 = N^2_{\beta,-\alpha^d_i}$$

and hence that $\frac{N_{\beta,\alpha_i}}{N_{\beta,-\alpha^d_i}} = \pm 1$. From this remark and (3.9), we conclude that $\mathfrak{m}^{10}_F(t)$ is generated by elements of the form

$$E_\alpha + \epsilon_\alpha e^{t\ell_\alpha t} E_{-\alpha^d} \ ,$$

where $\beta \in R^o_F$, $\alpha = \alpha_i + \beta$, $\alpha = \alpha_i + \beta$, $\alpha^d = \alpha^d_i + \beta$ and $\epsilon_\alpha = \frac{N_{\beta,\alpha_i}}{N_{\beta,-\alpha^d_i}}$. This concludes the proof of the claim.

For any root $\alpha \in R_F$, we call *CR-dual root of* α the root α^d so that $E_\alpha + \epsilon_\alpha e^{t\ell_\alpha t} E_{-\alpha^d} \in \mathfrak{m}^{10}(t)$.

We fix a positive root subsystem $R^+ \subset R$ so that $R'_+ = R^+ \cap (R \setminus (R^o \cup R^o_F \cup R'_F))$. Moreover, we decompose the set of roots R'_F into

$$R'_F = R^{(+)}_F \cup R^{(-)}_F$$

where

$$R^{(+)}_F = \{\alpha \in \ R'_F \ : \ E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \in \mathfrak{m}^{10} \ , \text{ with } \ell_\alpha = +1, +2 \ \}$$

$$R^{(-)}_F = \{\alpha \in \ R'_F \ : \ E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \in \mathfrak{m}^{10} \ , \text{ with } \ell_\alpha = -1, -2 \ \}$$

Using Table 1, one can check that in all cases

$$\mathfrak{m}^{10} = \mathrm{span}_{\mathbb{C}}\{ \ E_\alpha + \epsilon_\alpha e^{\ell_\alpha t} E_{-\alpha^d} \ , \ \alpha \in R^{(+)}_F \ \}$$

and that if $\alpha \in R^{(+)}_F$, then also the CR dual root $\alpha^d \in R^{(+)}_F$. We will denote by $\{\alpha_1, \alpha^d_1, \alpha_2, \alpha^d_2, \dots, \alpha_r, \alpha^d_r\}$ the set of roots in $R^{(+)}_F$ and by $\{\beta_1, \dots, \beta_s\}$ the roots in $R'_+ = R^+ \cap R'$.

Observe that the number of roots in $R^{(+)}_F$ is equal to $\frac{1}{2}\left(\dim_{\mathbb{R}} G_F/L_F - 1\right)$, where G_F/L_F is the Morimoto-Nagano space associated with the pair $(\mathfrak{g}_F, \mathfrak{l}_F)$.

Finally, we consider the following basis for $\mathbb{R}Z_{\mathcal{D}} + \mathfrak{m} \simeq T_{\eta_t} G \cdot \eta_t$. We set

$$F_0 = Z_{\mathcal{D}} \ ,$$

and, for any $1 \leq i \leq r$, we define the vectors F^+_i, F^-_i, G^+_i and G^-_i, as follows: in case $\{\alpha_i, \alpha^d_i\} \subset R^{(+)}_F$ is a pair of CR dual roots with $\alpha_i \neq \alpha^d_i$, we set

$$F^+_i = \frac{1}{\sqrt{2}}(F_{\alpha_i} + \epsilon_{\alpha_i} F_{\alpha^d_i}) \ , \quad F^-_i = \frac{1}{\sqrt{2}}(F_{\alpha_i} - \epsilon_{\alpha_i} F_{\alpha^d_i}) \ ,$$

$$G^+_i = \frac{1}{\sqrt{2}}(G_{\alpha_i} + \epsilon_{\alpha_i} G_{\alpha^d_i}) \ , \quad G^-_i = \frac{1}{\sqrt{2}}(G_{\alpha_i} - \epsilon_{\alpha_i} G_{\alpha^d_i}) \ , \tag{3.10}$$

where $\epsilon_{\alpha_i} = \pm 1$ is the integer which is defined in (3.8); in case $\{\alpha_i, \alpha^d_i\} \subset R^{(+)}_F$ is a pair of CR dual roots with $\alpha_i = \alpha^d_i$, we set

$$F^+_i = F_{\alpha_i} = \frac{E_{\alpha_i} - E_{-\alpha_i}}{\sqrt{2}} \ , \quad G^+_i = G_{\alpha_i} = i\frac{E_{\alpha_i} + E_{-\alpha_i}}{\sqrt{2}} \tag{3.10'}$$

and *we do not define the corresponding vectors* F^-_i *or* G^-_i. Finally, for any $1 \leq i \leq s = n - 1 - 2r$, we set

$$F'_i = F_{\beta_i} \ , \quad G'_i = G_{\beta_i} \ . \tag{3.11}$$

Note that in case r is odd, there is only one root $\alpha_i \in R_F^{(+)}$ such that $\alpha_i = \alpha_i^d$. When $\mathfrak{g}_F = \mathfrak{su}_2$, this root is also the *unique* root in $R_F^{(+)}$.

In case $\mathfrak{g}_F = \{0\}$, we set $F_0 = Z_{\mathcal{D}}$ and $F'_i = F_{\beta_i}$, $G'_i = G_{\beta_i}$ and we do not define the vector $F_i^{(\pm)}$ or $G_i^{(\pm)}$.

The basis $(F_0, F_k^{\pm}, F_j, G_k^{\pm}, G_j)$ for $\mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$, which we just defined, will be called *optimal basis associated with the optimal transversal curve* η. Notice that this basis is $\mathcal{B}$-orthonormal.

For simplicity of notation, we will often use the symbol F_k (resp. G_k) to denote any vector in the set $\{F_0, F_j^{\pm}, F'_j\}$ (resp. in $\{G_j^{\pm}, G'_j\}$). We will also denote by N_F the number of elements of the form $F_i^{\pm}$. Note that N_F is equal to half the real dimension of the holomorphic distribution of the Morimoto-Nagano space G_F/L_F.

For any odd integer $1 \leq 2k-1 \leq N_F$, we will assume that $F_{2k-1} = F_k^+$; for any even integer $2 \leq 2k \leq N_F$, we will assume $F_{2k} = F_k^-$. If N_F is odd, we denote by F_{N_F} the unique vector defined by (3.10'). We will also assume that $F_j = F'_{j-N_F}$ for any $N_F + 1 \leq j \leq n-1$.

In case M is a standard K-manifold, we assume that $N_F = 0$.

In the following lemma, we describe the action of the complex structure J_t in terms of an optimal basis.

Lemma 3.8. *Assume that η_t is an optimal transversal curve and let*

$$(F_0, F_k^{\pm}, F'_j, G_k^{\pm}, G'_j)$$

an associated optimal basis of $\mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$. Let also J_t be the complex structure of $\mathfrak{m}$ corresponding to the CR structure of a regular orbit $G \cdot \eta_t$.

Then $J_t F'_i = G'_i$ for any $1 \leq i \leq s = n - 1 - N_F$. Furthermore, if M is non-standard (i.e. $N_F > 0$) then:

(1) *if $1 \leq i \leq N_F$ and $\{\alpha_i, \alpha_i^d\}$, is a pair of CR-dual roots in $R_F^{(+)}$ with $\alpha_i \neq \alpha_i^d$ then*

$$J_t F_i^+ = -\coth(\ell_i t) G_i^+ \ , \quad J_t F_i^- = -\tanh(\ell_i t) G_i^- \ , \tag{3.12}$$

where ℓ_i is equal to 2 if $F_i^{\pm} \in [\mathfrak{m}_F, \mathfrak{m}_F]^{\mathbb{C}} \cap \mathfrak{m}_F^{\mathbb{C}}$ and it is equal to 1 otherwise;

(2) *if $1 \leq i \leq N_F$ and $\{\alpha_i, \alpha_i^d\}$ is a pair of CR-dual roots in $R_F^{(+)}$ with $\alpha_i = \alpha_i^d$, so that $F_i^+ = F_{\alpha_i}$, then*

$$J_t F_i^+ = -\coth(\ell_i t) G_i^+ \ , \tag{3.13}$$

where ℓ_i is equal to 2 if $F_i^{\pm} \in [\mathfrak{m}_F, \mathfrak{m}_F]^{\mathbb{C}} \cap \mathfrak{m}_F^{\mathbb{C}}$ and it is equal to 1 otherwise. Note that the case $\ell_i = 2$ may occur only if $\mathfrak{g}_F = \mathfrak{f}_4$ or $\mathfrak{sp}_n$ - see Table 1.

Proof. The first claim is an immediate consequence of Theorem 3.2 d) and the property of invariant complex structures on flag manifolds.

In order to prove (3.12), let us consider a pair $\{\alpha_i, \alpha_i^d\}$ of CR dual roots in R_F^+ with $\alpha_i \neq \alpha_i^d$; by the previous remarks, there exist two integers ℓ_i, ℓ_i^d,which are either $+1$ or $+2$, and two integers $\epsilon_{\alpha_i}, \epsilon_{\alpha_i^d} = \pm 1$, so that

$$E_{\alpha_i} + \epsilon_{\alpha_i} e^{2\ell_i t} E_{-\alpha_i^d} \ , \ E_{\alpha_i^d} + \epsilon_{\alpha_i^d} e^{2\ell_i^d t} E_{-\alpha_i} \in \mathfrak{m}_F^{10}(t)$$

for any $t \neq 0$.

By direct inspection of Table 1, one can check that the integers ℓ_i^d, ℓ_i are always equal. We claim that also $\epsilon_i = \epsilon_i^d$ for any CR dual pair $\{\alpha_i, \alpha_i^d\} \subset R_F^{(+)}$.

In fact, by conjugation, it follows that the following two vectors are in $\mathfrak{m}_F^{01}(t)$ for any $t \neq 0$:

$$E_{\alpha_i} + \frac{1}{\epsilon_{\alpha_i^d} e^{2\ell_i t}} E_{-\alpha_i^d} \ , \ E_{\alpha_i^d} + \frac{1}{\epsilon_{\alpha_i} e^{2\ell_i t}} E_{-\alpha_i} \in \mathfrak{m}_F^{01}(t) \ . \tag{3.14}$$

At this point, we recall that η_0 is a singular point for the G-action and that, by the structure theorems in [HS] (see also [AS]), the isotropy subalgebra $\mathfrak{g}_{\eta_0}$ contains the isotropy subalgebra $(\mathfrak{g}_F)_{\eta_0}$ of the non-complex singular G_F-orbit in M, which is a c.r.o.s.s.. In particular, one can check that $\dim_{\mathbb{R}}(\mathfrak{g}_F)_{\eta_0} = \dim_{\mathbb{R}} \mathfrak{l}_F + \dim_{\mathbb{C}} \mathfrak{m}_F^{01}$.

On the other hand, by Lemma 3.5 (2), we have that $(\mathfrak{g}_F)_{\eta_0} = \mathfrak{l}_F + \mathfrak{g} \cap \mathfrak{m}_F^{01}(0)$ and hence that

$$\dim_{\mathbb{R}}(\mathfrak{g} \cap \mathfrak{m}_F^{01}(0)) = \dim_{\mathbb{C}} \mathfrak{m}_F^{01}(0) \ . \tag{3.15}$$

Here, by $\mathfrak{m}_F^{01}(0)$ we denote the subspace which is obtained from Table 1, by setting the value of the parameter λ equal to $\lambda(0) = e^0 = 1$. Note that this subspace is *not* a Morimoto-Nagano subspace.

From (3.14), one can check that (3.15) occurs if and only if

$$\epsilon_{\alpha_i^d} = \epsilon_{\alpha_i} \tag{3.16}$$

for any pair of CR dual roots α_i, α_i^d. This proves the claim.

In all the following, we will use the notation $\epsilon_i = \epsilon_{\alpha_i} = \epsilon_{\alpha_i^d}$.

By some straightforward computation, it follows that, for any $t \neq 0$, the elements F_{α_i}, $F_{\alpha_i^d}$, G_{α_i} and $G_{\alpha_i^d}$ are equal to the following linear combinations of holomorphic and anti-holomorphic elements:

$$F_{\alpha_i} = \frac{1}{\sqrt{2}(1 - e^{4\ell_i t})} \left\{ \left[(E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i^d}) + \epsilon_i e^{2\ell_i t} (E_{\alpha_i^d} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i}) \right] + \right.$$

$$\left. + \left[-e^{4\ell_i t} (E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t}} E_{-\alpha_i^d}) - \epsilon_i e^{2\ell_i t} (E_{\alpha_i^d} + \frac{1}{\epsilon_i e^{2\ell_i t}} E_{-\alpha_i}) \right] \right\} \ ,$$

$$F_{\alpha_i^d} = \frac{1}{\sqrt{2}(1 - e^{4\ell_i t})} \left\{ \left[\epsilon_i e^{2\ell_i t} (E_{\alpha_i} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i^d}) + (E_{\alpha_i^d} + \epsilon_i e^{2\ell_i t} E_{-\alpha_i}) \right] - \right.$$

$$\left. - \left[e^{2\ell_i t} \epsilon_i (E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t}} E_{-\alpha_i^d}) + e^{4\ell_i t} (E_{\alpha_i^d} + \frac{1}{\epsilon_i e^{2\ell_i t}} E_{-\alpha_i}) \right] \right\} \ ,$$

$$G_{\alpha_i} = \frac{i}{\sqrt{2}(1-e^{4\ell_i t})}\left\{\left[(E_{\alpha_i} + \epsilon_i e^{2\ell_i t}E_{-\alpha_i^d}) - \epsilon_i e^{2\ell_i t}(E_{\alpha_i^d} + \epsilon_i e^{2\ell_i t}E_{-\alpha_i})\right] + \right.$$
$$\left. + \left[-e^{4\ell_i t}(E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t}}E_{-\alpha_i^d}) + \epsilon_i e^{2\ell_i t}(E_{\alpha_i^d} + \frac{1}{\epsilon_i e^{2\ell_i t}}E_{-\alpha_i})\right]\right\} ,$$
$$G_{\alpha_i^d} = \frac{i}{\sqrt{2}(1-e^{4\ell_i t})}\left\{\left[-\epsilon_i e^{2\ell_i t}(E_{\alpha_i} + \epsilon_i e^{2\ell_i t}E_{-\alpha_i^d}) + (E_{\alpha_i^d} + \epsilon_i e^{2\ell_i t}E_{-\alpha_i})\right] + \right.$$
$$\left. + \left[\epsilon_i e^{2\ell_i t}(E_{\alpha_i} + \frac{1}{\epsilon_i e^{2\ell_i t}}E_{-\alpha_i^d}) - e^{4\ell_i t}(E_{\alpha_i^d} + \frac{1}{\epsilon_i e^{2\ell_i t}}E_{-\alpha_i})\right]\right\} .$$

We then obtain that

$$J_t F_{\alpha_i} = \frac{1+e^{4\ell_i t}}{1-e^{4\ell_i t}}G_{\alpha_i} + \frac{2\epsilon_i e^{2\ell_i t}}{1-e^{4\ell_i t}}G_{\alpha_i^d} ,$$
$$J_t F_{\alpha_i^d} = \frac{2\epsilon_i e^{2\ell_i t}}{1-e^{4\ell_i t}}G_{\alpha_i} + \frac{1+e^{4\ell_i t}}{1-e^{4\ell_i t}}G_{\alpha_i^d} . \tag{3.17}$$

So, using the fact that $\epsilon_i^2 = 1$, we get $J_t F_i^+ = \frac{1+e^{2\ell_i t}}{1-e^{2\ell_i t}}G_i^+ = -\coth(\ell_i t)G_i^+$ and $J_t F_i^- = \frac{1-e^{2\ell_i t}}{1+e^{2\ell_i t}}G_i^- = -\tanh(\ell_i t)G_i^-$. The proof of (3.13) is similar. It suffices to observe that for any $t \neq 0$

$$F_i^+ = \frac{1}{\sqrt{2}(1-e^{4\ell_i t})}\left\{(1+e^{2\ell_i t})(E_{\alpha_i} + e^{2\ell_i t}E_{-\alpha_i}) - \right.$$
$$\left. - e^{2\ell_i t}(1+e^{2\ell_i t})(E_{\alpha_i} + e^{-2\ell_i t}E_{-\alpha_i})\right\} ,$$
$$G_i^+ = \frac{i}{\sqrt{2}(1-e^{4\ell_i t})}\left\{(1-e^{2\ell_i t})(E_{\alpha_i} + e^{2\ell_i t}E_{-\alpha_i}) + \right.$$
$$\left. + e^{2\ell_i t}(1-e^{2\ell_i t})(E_{\alpha_i} + e^{-2\ell_i t}E_{-\alpha_i})\right\} ,$$

and hence that $J_t F_i^+ = \frac{1+e^{2\ell_i t}}{1-e^{2\ell_i t}}G_i^+ = -\coth(\ell_i t)G_i^+$. □

4. The algebraic representatives of the Kähler and Ricci form of a K-manifold.

In this section we give a rigorous definition of the *algebraic representatives* of the Kähler form ω and the Ricci form ρ of a K-manifold. We will also prove Proposition 1.1.

Indeed, we will give the concept of 'algebraic representative' for any bounded, closed 2-form ϖ, which is defined on M_{reg} and which is G-invariant and J-invariant. Clearly, $\omega|_{M_{\text{reg}}}$ and $\rho|_{M_{\text{reg}}}$ belong to this class of 2-forms.

Let $\eta : \mathbb{R} \to M$ be an optimal transversal curve. Since $\mathfrak{g}$ is semisimple, for any G-invariant 2-form ϖ on M_{reg} there exists a unique $\text{ad}_{\mathfrak{l}}$-invariant element $F_{\varpi,t} \in \text{Hom}(\mathfrak{g},\mathfrak{g})$ such that:

$$\mathcal{B}(F_{\varpi,t}(X), Y) = \varpi_{\eta_t}(\hat{X}, \hat{Y}) , \qquad X, Y \in \mathfrak{g} ,\ t \neq 0 . \tag{4.1}$$

If ϖ is also closed, we have that for any $X, Y, W \in \mathfrak{g}$

$$0 = 3d\varpi(\hat{X}, \hat{Y}, \hat{W}) = \varpi(\hat{X}, [\hat{Y}, \hat{W}]) + \varpi(\hat{Y}, [\hat{W}, \hat{X}]) + \varpi(\hat{W}, [\hat{X}, \hat{Y}]) \ .$$

This implies that

$$F_{\varpi,t}([X, Y]), W) = [F_{\varpi,t}(X), Y] + [X, F_{\varpi,t}(Y)]$$

i.e. $F_{\varpi,t}$ is a derivation of $\mathfrak{g}$. Therefore, $F_{\varpi,t}$ is of the form

$$F_{\varpi,t} = \mathrm{ad}(Z_\varpi(t)) \tag{4.2}$$

for some $Z_\varpi(t) \in \mathfrak{g}$ and $\varpi_{\eta_t}(\hat{X}, \hat{Y}) = \mathcal{B}([Z_\varpi(t), X], Y) = \mathcal{B}(Z_\varpi(t), [X, Y])$. Note that since $F_{\varpi,t}$ is $\mathrm{ad}_{\mathfrak{l}}$-invariant, then $Z_\varpi(t) \in C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$, where $\mathfrak{a} = C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^\perp$.

We call the curve

$$Z_\varpi : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) + \mathfrak{a} \ , \tag{4.3}$$

the *algebraic representative of the 2-form ϖ along the optimal transversal curve η.*

By definition, if the algebraic representative $Z_\varpi(t)$ is given, it is possible to reconstruct the values of ϖ on any pair of vectors, which are tangent to the regular orbits $G \cdot \eta_t$. Actually, since for any point $\eta_t \in M_{\mathrm{reg}}$ we have that $J(T_{\eta_t}G) = T_{\eta_t}M$, it follows that one can evaluate ϖ on *any* pair of vectors in $T_{\eta_t}M$ if the value $\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, J\hat{Z}_{\mathcal{D}})$ is also given. However, in case ϖ is a closed form, the following Proposition shows that this last value can be recovered from the first derivative of the function $Z_\varpi(t)$.

Proposition 4.1. *Let (M, J, g) be a K-manifold acted on by the compact semisimple Lie group G. Let also $\eta_t = \exp(tiZ_{\mathcal{D}}) \cdot p_o$ be an optimal transversal curve and $Z_\varpi : \mathbb{R} \to \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$ the algebraic representative of a bounded, G-invariant, J-invariant closed 2-form ϖ along η. Then:*

(1) *if M is a standard K-manifold or a non-standard KO-manifold (i.e. if either $\mathfrak{a} = \mathbb{R}Z_{\mathcal{D}}$ or $\mathfrak{a} = \mathfrak{su}_2$ and M is standard), then there exists an element $I_\varpi \in \mathfrak{z}(\mathfrak{l})$ and a smooth function $f_\varpi : \mathbb{R} \to \mathbb{R}$ so that*

$$Z_\varpi(t) = f_\varpi(t) Z_{\mathcal{D}} + I_\varpi \ ; \tag{4.4}$$

(2) *if M is non-standard KE-manifold, then there exists a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}}$ and a root α of the corresponding root system, such that $Z_{\mathcal{D}} \in \mathbb{R}(iH_\alpha)$ and $\mathfrak{a} = \mathbb{R}Z_{\mathcal{D}} + \mathbb{R}F_\alpha + \mathbb{R}G_\alpha$; furthermore there exists an element $I_\varpi \in \mathfrak{z}(\mathfrak{l})$, a real number C_ϖ and a smooth function $f_\varpi : \mathbb{R} \to \mathbb{R}$ so that*

$$Z_\varpi(t) = f_\varpi(t) Z_{\mathcal{D}} + \frac{C_\varpi}{\cosh(t)} G_\alpha + I_\varpi \ . \tag{4.4'}$$

Conversely, if $Z_\varpi : \mathbb{R} \to C_{\mathfrak{g}}(\mathfrak{l})$ is a curve in $C_{\mathfrak{g}}(\mathfrak{l})$ of the form (4.4) or (4.4'), then there exists a unique closed J-invariant, G-invariant 2-form ϖ on M_{reg}, having $Z_\varpi(t)$ as algebraic representative; such 2-form is the unique J- and G-invariant form which verifies

$$\varpi_{\eta_t}(\hat{V}, \hat{W}) = \mathcal{B}(Z_\varpi(t), [V, W]) \ , \qquad \varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{Z}_{\mathcal{D}}) = -f'_\varpi(t) \mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}}) \ . \tag{4.5}$$

for any $V, W \in \mathfrak{m}$ *and any* $\eta_t \in M_{\text{reg}}$.

Proof. Let ϖ be a closed 2-form which is G-invariant and J-invariant and let $Z_\varpi(t)$ be the associated algebraic representative along η. Recall that $Z_\varpi(t) \in \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$. So, if the action is ordinary (i.e. $\mathfrak{a} = \mathbb{R}Z_{\mathcal{D}}$), $Z_\varpi(t)$ is of the form

$$Z_\varpi(t) = f_\varpi(t) Z_{\mathcal{D}} + I_\varpi(t) \ , \tag{4.6}$$

where the vector $I_\varpi(t) \in \mathfrak{z}(\mathfrak{l})$ may depend on t.

In case the action of G is extraordinary (that is $\mathfrak{a} = \mathfrak{su}_2$) by Lemma 2.2 in [PS], there exists a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}}$, such that $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\alpha + \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha}$ for some root α of the corresponding root system. By the arguments in the proof of Theorem 3.7, this Cartan subalgebra can be always chosen in such a way that $Z_{\mathcal{D}} \in \mathbb{R}(iH_\alpha)$ and hence that $\mathfrak{a} = \mathbb{R}Z_{\mathcal{D}} + \mathbb{R}F_\alpha + \mathbb{R}G_\alpha$.

Then the function $Z_\varpi(t)$ can be written as

$$Z_\varpi(t) = f_\varpi(t) Z_{\mathcal{D}} + g_\varpi(t) F_\alpha + h_\varpi(t) G_\alpha + I_\varpi(t) \tag{4.6'}$$

for some smooth real valued functions f_ϖ, g_ϖ and h_ϖ and some element $I_\varpi(t) \in \mathfrak{z}(\mathfrak{l})$.

We now want to show that, in case M is a non-standard KE-manifold, then $g_\varpi(t) \equiv 0$ and that $h_\varpi(t) = \frac{C_\varpi}{\cosh(t)}$ for some constant C_ϖ.

In fact, observe that if $Z_\varpi(t)$ is of the form (4.6') and if $Z_{\mathcal{D}}$ is as listed in Table 1 for $\mathfrak{g}_F = \mathfrak{su}_2$, then

$$\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha) = g_\varpi(t) \mathcal{B}(F_\alpha, [Z_{\mathcal{D}}, G_\alpha]) = -g_\varpi(t) \ ,$$

$$\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{F}_\alpha) = h_\varpi(t) \mathcal{B}(G_\alpha, [Z_{\mathcal{D}}, F_\alpha]) = h_\varpi(t) \ .$$

Consider now the facts that ϖ is closed, $\hat{G}_\alpha$ and $\hat{Z}_{\mathcal{D}}$ are holomorphic vector fields and $J\hat{Z}_{\mathcal{D}}|_{\eta_t} = \eta'_t$. It follows that g_ϖ verifies the following ordinary differential equation

$$\begin{aligned}
\left.\frac{dg_\varpi}{dt}\right|_{\eta_t} = &- \left.\frac{d}{dt}\varpi(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha)\right|_{\eta_t} = - \left. J\hat{Z}_{\mathcal{D}}\left(\varpi(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha)\right)\right|_{\eta_t} = \\
= \hat{G}_\alpha(\varpi(J\hat{Z}_{\mathcal{D}}, \hat{Z}_{\mathcal{D}}))\Big|_{\eta_t} &+ \hat{Z}_{\mathcal{D}}(\varpi(\hat{G}_\alpha, J\hat{Z}_{\mathcal{D}}))\Big|_{\eta_t} - \varpi_{\eta_t}([J\hat{Z}_{\mathcal{D}}, \hat{Z}_{\mathcal{D}}], \hat{G}_\alpha) - \\
&- \varpi_{\eta_t}([\hat{G}_\alpha, J\hat{Z}_{\mathcal{D}}], \hat{Z}_{\mathcal{D}}) - \varpi_{\eta_t}([\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha], J\hat{Z}_{\mathcal{D}})\Big] = \\
&= \varpi_{\eta_t}([\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha], J\hat{Z}_{\mathcal{D}}) = -\varpi_{\eta_t}(\widehat{[Z_{\mathcal{D}}, G_\alpha]}, J\hat{Z}_{\mathcal{D}}) = \\
= -\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, J\hat{F}_\alpha) &= \coth(t)\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha) = -\coth(t) g_\varpi(t) \ .
\end{aligned} \tag{4.7}$$

We claim that this implies

$$g_\varpi(t) \equiv 0 \ . \tag{4.8}$$

In fact, if we assume that $g_\varpi(t)$ does not vanish identically, integrating the above equation, we have that $g_\varpi(t) = \frac{C}{|\sinh(t)|}$ for some $C \neq 0$ and hence with a singularity at $t = 0$. But this contradicts the fact that ϖ is a bounded 2-form.

With a similar argument, we have that $h_\varpi(t)$ verifies the differential equation

$$\left.\frac{dh_\varpi}{dt}\right|_{\eta_t} = -\tanh(t)h_\varpi(t) \; ;$$

by integration this gives

$$h_\varpi(t) = \frac{C_\varpi}{\cosh(t)} \tag{4.9}$$

for some constant C_ϖ.

We show now that, in case M is a standard KE-manifold, then $Z_\varpi(t)$ is of the form (4.4). In fact, even if a priori $Z_\varpi(t)$ is of the form (4.6'), from Lemma 3.8 and the same arguments for proving (4.7), we obtain that

$$\left.\frac{dg_\varpi}{dt}\right|_{\eta_t} = -\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, J\hat{F}_\alpha) = -\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha) = g_\varpi(t) \; . \tag{4.10}$$

This implies that $g_\varpi(t) = Ae^t$ for some constant A. On the other hand, if $A \neq 0$, it would follow that $\lim_{t\to\infty} |\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha)| = \lim_{t\to\infty} |g_\varpi(t)| = +\infty$, which is impossible since $\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}}, \hat{G}_\alpha)$ is bounded. Hence $g_\varpi(t) \equiv 0$.

A similar argument proves that $h_\varpi(t) \equiv 0$.

In order to conclude the proof, it remains to show that in all cases the element $I_\varpi(t)$ is independent on t and that $\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, Z_{\mathcal{D}}) = -f'_\varpi(t)\mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}})$ for any t. We will prove these two facts only for the case $\mathfrak{a} \simeq \mathfrak{sl}_2(\mathbb{R})$ and M non-standard, since the proof in all other cases is similar.

Consider two elements $V, W \in \mathfrak{g}$. Since ϖ is closed we have that

$$\begin{aligned}
0 = 3d\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{V}, \hat{W}) = \\
= J\hat{Z}_{\mathcal{D}}(\varpi_{\eta_t}(\hat{V}, \hat{W})) - \hat{V}(\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{W})) + W(\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{V})) - \\
-\varpi_{\eta_t}([J\hat{Z}_{\mathcal{D}}, \hat{V}], \hat{W}) + \varpi_{\eta_t}([J\hat{Z}_{\mathcal{D}}, \hat{W}], \hat{V}) - \varpi_{\eta_t}([\hat{V}, \hat{W}], J\hat{Z}_{\mathcal{D}}) = \\
= J\hat{Z}_{\mathcal{D}}|_{\eta_t}(\varpi(\hat{V}, \hat{W})) - \varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, [\hat{V}, \hat{W}]) = \\
= \left.\frac{d}{dt}\left(\mathcal{B}(Z_\varpi, [V, W])\right)\right|_t + \varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \widehat{[V, W]}) \; .
\end{aligned} \tag{4.11}$$

On the other hand, we have the following orthogonal decomposition of the element $[V, W]$:

$$[V, W] = \frac{\mathcal{B}(Z_{\mathcal{D}}, [V, W])}{\mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}})} Z_{\mathcal{D}} - \mathcal{B}(F_\alpha, [V, W])F_\alpha - \mathcal{B}(G_\alpha, [V, W])G_\alpha +$$

$$+[V, W]_{(\mathfrak{l}+\mathfrak{a})^\perp} + [V, W]_{\mathfrak{l}} \; ,$$

where $[V, W]_{\mathfrak{l}}$ and $[V, W]_{(\mathfrak{l}+\mathfrak{a})^\perp}$ are the orthogonal projections of $[V, W]$ into $\mathfrak{l}$ and $(\mathfrak{l}+\mathfrak{a})^\perp$, respectively. Then

$$\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \widehat{[V, W]}) = \frac{\mathcal{B}(Z_{\mathcal{D}}, [V, W])}{\mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}})} \varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{Z}_{\mathcal{D}}) - \mathcal{B}(F_\alpha, [V, W])\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}}, \hat{F}_\alpha) -$$

$$-\mathcal{B}(G_\alpha,[V,W])\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{G}_\alpha)+\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\widehat{[V,W]}_{(\mathfrak{l}+\mathfrak{a})^\perp})=$$

$$=\frac{\mathcal{B}(Z_{\mathcal{D}},[V,W])}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{Z}_{\mathcal{D}})+\mathcal{B}(F_\alpha,[V,W])\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}},J\hat{F}_\alpha)+$$

$$+\mathcal{B}(G_\alpha,[V,W])\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}},J\hat{G}_\alpha)-\varpi_{\eta_t}(\hat{Z}_{\mathcal{D}},J\widehat{[V,W]}_{(\mathfrak{l}+\mathfrak{a})^\perp})=$$

$$=\frac{\mathcal{B}(Z_{\mathcal{D}},[V,W])}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{Z}_{\mathcal{D}})+\mathcal{B}(G_\alpha,[V,W])\frac{C_\varpi\tanh(t)}{\cosh(t)}-$$

$$-\mathcal{B}(Z_\varpi(t),\left[Z_{\mathcal{D}},J_{\eta_t}([V,W]_{(\mathfrak{l}+\mathfrak{a})^\perp})\right])=$$

$$=\mathcal{B}(\left\{\frac{\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{Z}_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}Z_{\mathcal{D}}-h'_\varpi(t)G_\alpha\right\},[V,W])\ .$$

Therefore (4.11) becomes

$$\mathcal{B}\left(\left\{f'_\varpi(t)+\frac{\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{Z}_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}\right\}Z_{\mathcal{D}}+\frac{dI_\varpi}{dt},[V,W]\right)=0\ .$$

Since V,W are arbitrary and $\frac{dI_\varpi}{dt}\in\mathfrak{z}(l)\subset(Z_{\mathcal{D}})^\perp$, it implies

$$f'_\varpi(t)=-\frac{\varpi_{\eta_t}(J\hat{Z}_{\mathcal{D}},\hat{Z}_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}\ ,\qquad \frac{dI_\varpi}{dt}\equiv 0\ ,$$

as we needed to prove. □

We conclude this section, with the following corollary which gives a geometric interpretation of the optimal bases (see also §1).

Corollary 4.2. *Let (M,J,g) be a K-manifold and let (F_i,G_i) be an optimal basis along an optimal transversal curve $\eta_t=\exp(tiZ)\cdot p_o$. For any $\eta_t\in M_{\text{reg}}$, denote by $\mathcal{F}_t=(e_0,e_1,\dots,e_n)_t$ the following holomorphic frame in $T^{\mathbb{C}}_{\eta_t}M$:*

$$e_0=\hat{F}_0|_{\eta_t}-iJ\hat{F}_0|_{\eta_t}=\hat{Z}|_{\eta_t}-iJ\hat{Z}|_{\eta_t}\ ,\quad e_i=\hat{F}_i|_{\eta_t}-iJ\hat{F}|_{\eta_t}\quad i\geq 1\ .$$

Then,

1. *if M is a KO-manifold or a standard KE-manifold, then the holomorphic frames $\mathcal{F}_t$ are orthogonal w.r.t. any G-invariant Kähler metric g on M;*
2. *if M is a non-standard KE-manifold, then the holomorphic frames $\mathcal{F}_t$ are orthogonal w.r.t. any G-invariant Kähler metric g on M, whose associated algebraic representative $Z_\omega(t)$ has vanishing coefficient $C_\omega=0$ (see Proposition 4.1 for the definition of C_ω)*

Proof. It is a direct consequence of definitions and Proposition 4.1. □

5. The Ricci tensor of a K-manifold.

From the results of §4, the Ricci form ρ can be completely recovered from the algebraic representative $Z_\rho(t)$ along an optimal transversal curve η_t. On the other hand, using a few known properties of flag manifolds, the reader can check that the curve $Z_\rho(t) \in \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$ is uniquely determined by the 1-parameter family of quadratic forms Q^r on $\mathfrak{m}$ given by

$$Q^r_t : \mathfrak{m} \to \mathbb{R} \ , \qquad Q^r_t(E) = r_{\eta_t}(\hat{E}, \hat{E}) \qquad \Big(= -\rho_{\eta_t}(\hat{E}, \hat{E}) = -\mathcal{B}(Z_\rho(t), [E, J_t E]) \Big) \ .$$

Since $\mathfrak{m}$ corresponds to the subspace $\mathcal{D}_{\eta_t} \subset T_{\eta_t} G \cdot \eta_t$, this means that *for any Kähler metric ω, the corresponding the Ricci tensor r is uniquely determined by its restrictions $r|_{\mathcal{D}_t \times \mathcal{D}_t}$ on the holomorphic tangent spaces $\mathcal{D}_t$ of the regular orbits $G \cdot \eta_t$.*

The expression for the restrictions $r|_{\mathcal{D}_t \times \mathcal{D}_t}$ in terms of the algebraic representative $Z_\omega(t)$ of the Kähler form ω is given in the following Theorem.

Theorem 5.1. *Let (M, J, g) be a K-manifold and $\eta_t = \exp(tiZ_{\mathcal{D}}) \cdot p_o$ be an optimal transversal curve. Using the same notation of §3, let also $(F_i, G_i) = (F_0, F^\pm_k, G^\pm_k, F'_j, G'_j)$ be an optimal basis for $\mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$; finally, for any $1 \leq j \leq N_F$ let ℓ_j be the integer which appear in (3.12) for the expression of $J_t F_i$ and for any $N_F + 1 \leq k \leq n-1$ let β_k be the root so that $F_k = F_{\beta_k}$.*

Then, for any $\eta_t \in M_{\text{reg}}$ and for any element $E \in \mathfrak{m}$

$$\rho_{\eta_t}(\hat{E}, J\hat{E}) = A_E(t) \left\{ \frac{1}{2} h'(t) - \sum_{i=1}^{N_F} \tanh^{(-1)^{i+1}}(\ell_i t)\ell_i + \sum_{j=N_F+1}^{n-1} \beta_j(iZ_{\mathcal{D}}) \right\} + B_E(t) \tag{5.1}$$

where

$$h(t) = \log(\omega^n(\hat{F}_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1})|_{\eta_t}) \ , \tag{5.2}$$

$$A_E(t) = \frac{\mathcal{B}([E, J_t E], Z_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}})} \ , \tag{5.3}$$

$$\begin{aligned} B_E(t) = & - \sum_{i=1}^{N_F} \tanh^{(-1)^{i+1}}(\ell_i t) \mathcal{B}([E, J_t E]_{\mathfrak{l}+\mathfrak{m}}, [F_i, G_i]_{\mathfrak{l}+\mathfrak{m}}) + \\ & + \sum_{j=N_F+1}^{n-1} \mathcal{B}\left(iH_{\beta_j}, [E, J_t E]_{\mathfrak{z}(\mathfrak{l})}\right) \ , \end{aligned} \tag{5.4}$$

and where, for any $X \in \mathfrak{g}$, we denote by $X_{\mathfrak{l}+\mathfrak{m}}$ (resp. $X_{\mathfrak{z}(\mathfrak{l})}$) the projection parallel to $(\mathfrak{l} + \mathfrak{m})^\perp = \mathbb{R}Z_{\mathcal{D}}$ (resp. to $\mathfrak{z}(\mathfrak{l})^\perp$) of X into $\mathfrak{l} + \mathfrak{m}$ (resp. into $\mathfrak{z}(\mathfrak{l})$).

Proof. Let J_t be the complex structure on $\mathfrak{m}$ induced by the complex structure J of M. For any $E \in \mathfrak{m}$ and any point η_t, we may clearly write that $\rho_{\eta_t}(\hat{E}, J\hat{E}) = \rho_{\eta_t}(\hat{E}, \widehat{J_t E})$ and hence, by Koszul's formula (see [Ko], [Be]),

$$\rho_{\eta_t}(\hat{E}, J\hat{E}) = \frac{1}{2} \frac{\left(\mathcal{L}_{J\widehat{[E, J_t E]}} \omega^n\right)_{\eta_t} (\hat{F}_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1})}{\omega^n_{\eta_t}(\hat{F}_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \ldots, JF_{n-1})} \tag{5.5}$$

(note that the definition we adopt here for the Ricci form ρ is opposite in sign to the definition used in [Be]).

Recall that for any $Y \in \mathfrak{g}$, we may write

$$\hat{Y}|_{\eta(t)} = \sum_{i\geq 0} \lambda_i \hat{F}_i|_{\eta(t)} + \sum_{i\geq 1} \mu_i J\hat{F}_i|_{\eta(t)} \ ,$$

where

$$\lambda_i = \frac{\mathcal{B}(Y, F_i)}{\mathcal{B}(F_i, F_i)} \ , \qquad \mu_i = \frac{\mathcal{B}(Y, J_t F_i)}{\mathcal{B}(J_t F_i, J_t F_i)} \ .$$

Hence, for any i

$$[J\widehat{[E, J_t E]}, \hat{F}_i]_{\eta_t} = -J[\widehat{[E, J_t E]}, F_i]_{\eta_t} =$$

$$= -\sum_{j\geq 0} \frac{\mathcal{B}([[E, J_t E], F_i], F_j)}{\mathcal{B}(F_j, F_j)} J\hat{F}_j|_{\eta(t)} + \sum_{j\geq 1} \frac{\mathcal{B}([[E, J_t E], F_i], J_t F_j)}{\mathcal{B}(J_t F_j, J_t F_j)} \hat{F}_j|_{\eta(t)} =$$

$$= -\sum_{j\geq 0} \frac{\mathcal{B}([E, J_t E], [F_i, F_j])}{\mathcal{B}(F_j, F_j)} J\hat{F}_j|_{\eta(t)} + \sum_{j\geq 1} \frac{\mathcal{B}([E, J_t E], [F_i, J_t F_j])}{\mathcal{B}(J_t F_j, J_t F_j)} \hat{F}_j|_{\eta(t)} \ , \qquad (5.6)$$

$$[J\widehat{[E, J_t E]}, J\hat{F}_i]_{\eta_t} = [\widehat{[E, J_t E]}, F_i]_{\eta_t} =$$

$$= \sum_{j\geq 0} \frac{\mathcal{B}([E, J_t E], [F_i, F_j])}{\mathcal{B}(F_j, F_j)} \hat{F}_j|_{\eta(t)} + \sum_{j\geq 1} \frac{\mathcal{B}([E, J_t E], [F_i, J_t F_j])}{\mathcal{B}(J_t F_j, J_t F_j)} J\hat{F}_j|_{\eta(t)} \ . \qquad (5.7)$$

Therefore, if we denote $h(t) = \log(\omega^n(\hat{F}_0, J\hat{F}_0, \hat{F}_1, J\hat{F}_1, \dots, JF_{n-1})|_{\eta_t})$, then, after some straightforward computations, (5.5) becomes

$$\rho_{\eta_t}(\hat{E}, \widehat{J_t E}) = \frac{1}{2} J\widehat{[E, J_t E]}(h)|_{\eta_t} - \sum_{i\geq 1}^{n-1} \frac{\mathcal{B}([E, J_t E], [F_i, J_t F_i])}{\mathcal{B}(J_t F_i, J_t F_i)} \ . \qquad (5.8)$$

We claim that

$$J\widehat{[E, J_t E]}(h)|_{\eta_t} = A_E(t) h'_t \ . \qquad (5.9)$$

In fact, for any $X \in \mathfrak{g}$

$$\hat{X}(\omega(\hat{F}_0, J\hat{F}_0, \dots, J\hat{F}_{n-1})|_{\eta_t} =$$

$$= -\omega_{\eta_t}(\widehat{[X, F_0]}, J\hat{F}_0, \dots, J\hat{F}_{n-1}) - \omega(F_0, J\widehat{[X, F_0]}, \dots, J\hat{F}_{n-1}) - \dots = 0 \ . \qquad (5.10)$$

On the other hand,

$$J\widehat{[E, J_t E]}|_{\eta_t} = \frac{\mathcal{B}([E, J_t E], Z_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}}, Z_{\mathcal{D}})} J\hat{Z}_{\mathcal{D}}|_{\eta_t} + J\hat{X}_{\eta_t} = A_E(t) J\hat{Z}_{\mathcal{D}}|_{\eta_t} + \widehat{J_t X}_{\eta_t} \qquad (5.11)$$

for some some $X \in \mathfrak{m}$. From (5.11) and (5.10) and the fact that $J\hat{Z}_{\mathcal{D}}|_{\eta_t} = \eta'_t$, we immediately obtain (5.9).

Let us now prove that

$$\sum_{i\geq 1}^{n-1} \frac{\mathcal{B}([E,J_tE],[F_i,J_tF_i])}{\mathcal{B}(J_tF_i,J_tF_i)} = A_E \left\{ \sum_{i=1}^{N_F} \tanh^{(-1)^{i+1}}(\ell_i t)\ell_i - \sum_{j=N_F+1}^{n-1} \beta_i(iZ_{\mathcal{D}}) \right\} - B_E \tag{5.12}$$

First of all, observe that from definitions, for any $1 \leq k \leq N_F$ we have that, for any case of Table 1, when $\alpha_k \neq \alpha_k^d$,

$$\mathcal{B}(Z_{\mathcal{D}},[F_k,G_k]) = \frac{1}{2}\mathcal{B}(Z_{\mathcal{D}},[F_{\alpha_k} + (-1)^{k+1}\epsilon_k F_{\alpha_k^d}, G_{\alpha_k} + (-1)^{k+1}\epsilon_k G_{\alpha_k^d}]) =$$

$$= \frac{i}{2}\mathcal{B}(Z_{\mathcal{D}}, H_{\alpha_k} + H_{\alpha_k^d}) = \ell_k \ , \tag{5.13}$$

and, when $\alpha_k = \alpha_k^d$,

$$\mathcal{B}(Z_{\mathcal{D}},[F_k,G_k]) = \mathcal{B}(Z_{\mathcal{D}},[F_{\alpha_k},G_{\alpha_k}]) = \mathcal{B}(Z_{\mathcal{D}}, iH_{\alpha_k}) = \ell_k \ . \tag{5.13'}$$

Similarly, for any $N_F + 1 \leq j \leq n-1$

$$\mathcal{B}(Z_{\mathcal{D}},[F_j,G_j]) = \mathcal{B}(Z_{\mathcal{D}}, iH_{\beta_j}) = \beta_j(iZ_{\mathcal{D}}) \ . \tag{5.14}$$

So, using (5.13), (5.13'), (5.14) and the fact that $\mathcal{B}(F_i,F_i) = \mathcal{B}(G_i,G_i) = -1$ for any $1 \leq i \leq n-1$, we obtain that for $1 \leq k \leq N_F$,

$$\frac{\mathcal{B}([E,J_tE],[F_k,J_tF_k])}{\mathcal{B}(J_tF_k,J_tF_k)} = \tanh^{(-1)^{k+1}}(\ell_k t)\left(\mathcal{B}(Z_{\mathcal{D}},[F_k,G_k])\frac{\mathcal{B}([E,J_tE],Z_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})} + \right.$$

$$\left. + \qquad \mathcal{B}([E,J_tE]_{\mathfrak{l}+\mathfrak{m}},[F_k,G_k]_{\mathfrak{l}+\mathfrak{m}})\right) =$$

$$= \tanh^{(-1)^{k+1}}(\ell_k t)\left[A_E(t)\ell_k + \mathcal{B}([E,J_tE]_{\mathfrak{l}+\mathfrak{m}},[F_k,G_k]_{\mathfrak{l}+\mathfrak{m}})\right] \ , \tag{5.15}$$

and for any $N_F + 1 \leq j \leq N$

$$\frac{\mathcal{B}([E,J_tE],[F_j,J_tF_j])}{\mathcal{B}(J_tF_j,J_tF_j)} =$$

$$= -\frac{\mathcal{B}([E,J_tE],Z_{\mathcal{D}})}{\mathcal{B}(Z_{\mathcal{D}},Z_{\mathcal{D}})}\mathcal{B}(Z_{\mathcal{D}},[F_j,G_j]) - \mathcal{B}([E,J_tE]_{\mathfrak{l}+\mathfrak{m}}[F_j,G_j]_{\mathfrak{l}+\mathfrak{m}}) =$$

$$= -A_E\beta_j(iZ_{\mathcal{D}}) - \mathcal{B}(iH_{\beta_j},[E,J_tE]_{\mathfrak{z}(\mathfrak{l})}) \ . \tag{5.16}$$

From (5.15) and (5.16), we immediately obtain (5.12) and from (5.8) this concludes the proof. □

The expressions for the functions $A_E(t)$ and $B_E(t)$ simplify considerably if one assumes that E is an element of the optimal basis. Such expressions are given in the following conclusive proposition.

Proposition 5.2. *Let (F_i, G_i) be an optimal basis along an optimal transversal curve η_t of a K-manifold M. For any $1 \leq i \leq N_F$, let ℓ_i be as in Theorem 5.1 and denote by $\{\alpha_i, \alpha_i^d\} \subset R_F^{(+)}$ the pair of CR-dual roots, such that $F_i = \frac{1}{\sqrt{2}}(F_{\alpha_i} \pm \epsilon_i F_{\alpha_i^d})$ or $F_i = F_{\alpha_i}$, in case $\alpha_i = \alpha_i^d$; also, for any $N_F + 1 \leq j \leq n-1$, denote by $\beta_j \in R'_+$ the root such that $F_j = F_{\beta_j}$. Finally, let $A_E(t)$ and $B_E(t)$ be as defined in Theorem 5.1 and let us denote by*

$$Z^\kappa = \sum_{k=N_F+1}^{n-1} iH_{\beta_k} \ . \tag{5.17}$$

(1) *If $E = F_i$ for some $1 \leq i \leq N_F$, then*

$$A_{F_i}(t) = -\frac{\ell_i \tanh^{(-1)^i}(\ell_i t)}{\mathcal{B}(Z_\mathcal{D}, Z_\mathcal{D})} \ ; \tag{5.18}$$

and

$$B_{F_i}(t) = -\frac{\ell_i \tanh^{(-1)^i}(\ell_i t)}{\mathcal{B}(Z_\mathcal{D}, Z_\mathcal{D})}\mathcal{B}(Z^\kappa, Z_\mathcal{D}) +$$

$$+ \tanh^{(-1)^i}(\ell_i t) \left(\sum_{j=1}^{N_F} \tanh^{(-1)^{j+1}}(\ell_j t) \mathcal{B}([F_i, G_i]_{\mathfrak{l}+\mathfrak{m}}, [F_j, G_j]_{\mathfrak{l}+\mathfrak{m}}) \right) \ , \tag{5.19}$$

(2) *If $E = F_i$ for some $N_F + 1 \leq i \leq n-1$, then*

$$A_{F_i}(t) = \frac{\mathcal{B}(Z_\mathcal{D}, iH_{\beta_i})}{\mathcal{B}(Z_\mathcal{D}, Z_\mathcal{D})} \ , \quad B_{F_i}(t) = \mathcal{B}(Z^\kappa, iH_{\beta_i}) \ . \tag{5.20}$$

Proof. Formulae (5.18) and (5.19) are immediate consequences of definitions and of (5.13), (5.13') and (5.14). Formula (5.20) can be checked using the fact that $[F_{\beta_i}, J_t F_{\beta_i}] = [F_{\beta_i}, G_{\beta_i}] = iH_{\beta_i}$ for any $N_F + 1 \leq i \leq n-1$, from properties of the Lie brackets $[F_i, G_i]$, with $1 \leq i \leq N_F$, which can be derived from Table 1, and from the fact that $\mathbb{R}Z_\mathcal{D} \subset [\mathfrak{m}', \mathfrak{m}']^\perp$. $\square$

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